Advised Introduction to

Rocket Science (Including Crashes)

for Kerbals

Contents

Introduction

> “That’s one small step for a man, one giant leap for mankind.”
>
> – Neil Armstrong, first human being on the Moon

0.1 About this document

0.2 Kerbal Space Program

Kerbal Space Program [1] is a rocket simulation game; it lets you build, launch and pilot a rocket to put up satellites, send probes, land rovers, and have Kerbals do SCIENCE.

A big advantage a KSP player has over actual rocket crafters is that she can have rockets exploding without having to worry about consequences. Half the fun of the game comes from finding out why your next rocket will fail; thus, I urge you roll rockets to the launchpad early and not bother thinking of every little thing that could go wrong.

Airsick will not cover the initiation to the game, since the interface is specific to the game and prone to change. The game offers an intuitive way to assemble rockets and has a handy interface to control the flight. Rather, it will expose, explain and detail real principles of physics that the game mimics. This information can be read simply for curiosity, or can serve to design more precise launches in the game. When possible, example values will be taken from reality or KSP.

? KSP is a propietary software. You can play the demo for free, but will be limited to an older version with only a few parts available for building. The demo is way harder and the complete game way richer.

! This document is intended to provide information both for beginners and actual physicists. The first few chapters are an introduction to the elemen-

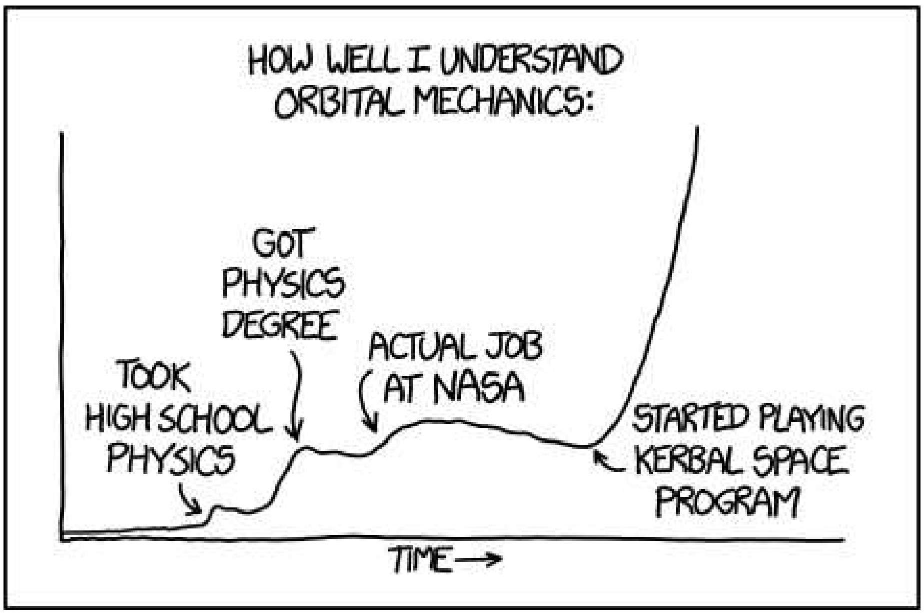


Figure 1: Mouseover text reads ”To be fair, my job at NASA was working on robots and didn’t actually involve any orbital mechanics. The small positive slope over that period is because it turns out that if you hang around at NASA, you get in a lot of conversations about space.” [15]

tary concepts used within this book. In every chapters accurate derivations and complete proofs are developed wherevever needed. If you are not interested in the specifics of a demonstration or have already some knowledge of basics physics, feel free to skip irrelevant parts.

0.3 Credits

This document makes extensive use of pictures, diagrams and various illustrations. Most figures were generated with the file from LaTeXusing the Tikz package. When an external image was included as a figure, a reference to the source is linked in the caption (e.g. [15]).

Sources for the other illustrations:

- front cover [2]
- chapter banners [3]
- back cover [1]

Chapter 1

Basics

“Imagination will often carry us to worlds that never were. But without it we go nowhere.”

– Carl Sagan, popular atrophysicist

1.1 Units

1.1.1 Dimensions

A number can be used to count items or measure different concepts. For instance, x might be refering to a duration, an angle, a mass, a pressure, a distance, a speed, an acceleration, a force, etc. The concept x is referring to is called the **dimension** of x.

There is usually several ways to measure a similar concept. For instance, a distance might be measured in either meters, feet, stadiums, etc. To avoid confusion, we agreed on which units should be prefered; those are called SI Units (for *Système International*, which is French for International System).

Below is a summary of some dimensions, the corresponding SI unit, and other common units:

Dimension	SI unit	Other units
angle	radian (rad)	turn (-), degree (°)
duration	second (s)	hour (h), day (d), year (y)
distance	meter (m)	foot (ft), mile (mi), light-year (ly)
speed	meter per second (m/s)	mile per hour (mph), knot (kn)
mass	kilogram (kg)	ton (t), pound (lb)
pressure	pascal (Pa)	atmosphere (atm), bar (bar)

A light year is the distance that a particle of light can travel in a year. For comparison, it takes light a little bit more than eight minutes (8 min) to get from the Sun to the Earth, meaning that the Sun is 8 light-minutes away from the Earth. Kerbin is about 45 light-seconds away from Kerbol.

1.1.2 Prefixes

When considering different scales, it is practical to use different units. Using a same unit when traveling or when describing a stamp would force us to use tiny and huge numbers, making it harder to build an intuition.

Such units have been hinted above. For instance, the ångström is used for the size of atoms and molecules; the parsec is used for interstellar distances. Creating a new unit for each use case is cumberstone and makes it harder to concialiate intersecting situations.

A simpler approach is to use **prefixes**. The idea is to easily create new units out of a basic one. That way, a **kilo**meter is 1,000 meters and we can simply write 355 km rather than 355,000 m.

Here are the most common prefixes:

kilo- (k-)	mega- (M-)	giga- (G-)	tera- (T-)
10^3	10^6	10^9	10^{12}

There are also prefixes to decrease the value of an unit:

milli- (m-)	micro- (μ-)	nano- (n-)	pico- (p-)
10^{-3}	10^{-6}	10^{-9}	10^{-12}

1.1.3 Conversion

We sometimes need to switch the unit used for a measure. For example, let us convert $50\ km/h$ to SI units. We know that $km = 1000\ m$ and $h = 3600\ s$ so:

$$50\ km/h = 50(1000\ m)/(3600\ s) = 50 \times 1000/3600\ m/s = 14\ m/s$$

This particular example shows how easy it is to include units in computations. As we will see below, having the units is useful when considering more complex expressions.

1.1.4 Addition (and substraction)

An addition involves two measures of the same *dimension*. For example, let us assume we have a distance x defined as follows:

$$x = 2\ ly + 4,730\ Tm$$

Since we know that that $ly = 9,4607\ Tm$, we can replace it in the expression:

$$\begin{aligned} x &= 2ly + 4,730\ Tm \\ &= 2(9,461\ Tm) + 4,730\ Tm \\ &= (18,922 + 4,730)Tm \\ &= 23,652\ Tm \end{aligned}$$

Conversely, we could also have said that $Tm = 1/9,461\ ly$ and then:

$$\begin{aligned} x &= 2\ ly + 4,730\ Tm \\ &= 2\ ly + (4,730/9,461)ly \\ &= (2 + 0.5)ly \\ &= 2.5\ ly \end{aligned}$$

Of course, the two ways are equivalent and we can check that $2.5\ ly = 23,652\ Tm$.

1.1.5 Multiplication (and division)

We can create new units pretty easily. For example, if some object travels a distance $d = 15\ m$ in a duration $t = 3\ s$, we can define the velocity $v = d/t = 15\ m/3\ s = 5\ m/s$. Conversely, assume the object has traveled at a velocity $50\ km/h$ for a duration $30\ s$; then, the distance it has gone through is:

$$d = v \times t = (50\ km/h) \times (30\ s) = 14\ m/s \times 30\ s = (14 \times 30)(m/\cancel{s} \times \cancel{s}) = 417\ m$$

By doing the operations on both the numerical values and the units, we know what our result is: in this case, it's a distance, and it is expressed in meters.

1.2 Functions

Let us consider a car C moving along a straight road at a speed of v (e.g. $50\ km/h$). The position of the car, C, can be determined by the distance from C to an arbitrary fix point O (the **origin**). We will note this distance x and we have thus $x = OC$.

We will measure time t as the delay since the car was at the origin ($C = O$).

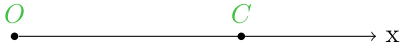


Figure 1.1: C is moving towards the right at speed v

Say we want to follow the evolution of the position of the car as time passes by. In other words, we are interested in knowing x as a **function** of t. We know that, at time t, we have $x = v \times t$. We write it:

$$x(t) = v \times t$$

This notation gives us a general formula to compute x for any given value of t. For example, if we want to know the position of the car after one hour:

$$x(1\ h) = 50\ km/h \times 1\ h = 50\ km$$

As another example, it is known that the intensity of the light emitted by a star decreases proportionnaly to the square of the distance to the star. This can be written as:

$$L(r) = \frac{C}{r^2}$$

where C is some **constant** value (i.e. independent from r) which is to be determined experimentally.

1.3 Derivatives

1.3.1 Definition

Assume we know the position $x(t)$ of the car for any instant t and we want to determine the velocity of the car at a given instant t_0.

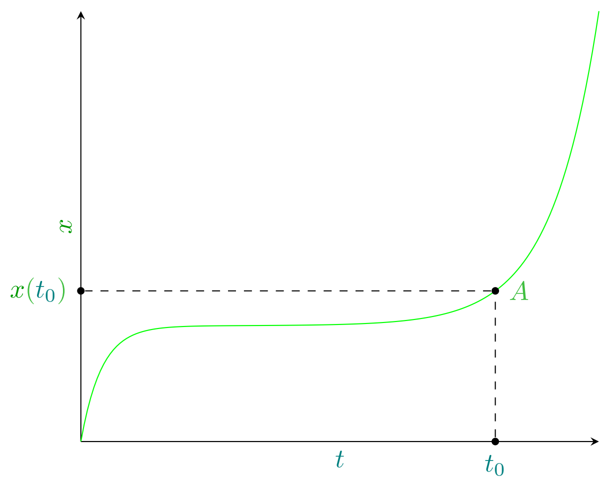


Figure 1.2: The horizontal axis represents the passage of time, the vertical axis the position. The curve shows the position at every instant. For instance, at instant t_0, the position is $x(t_0)$, which corresponds to the point A.

The velocity is the **variation** of position through time. Thus, to know how fast the car is going at time t_0, we need to look at the position of the car at two different instants. We already have t_0; let us also consider $t_0 + h$ for some arbitrary value h.

The difference in position between instants t_0 and $t_0 + h$ is thus $x(t_0 + h) - x(t_0)$; a shorter notation for this is $\Delta x(t_0)$. The value h is not shown because it has no importance in itself. The Greek letter Δ ("delta", equivalent of d), is generally used to denote a **d**ifference (here, the difference in position).

Notice that, the bigger h, the bigger we expect this difference to be: the longer the delay, the longer the car moved. To compensate for this, we will divide by how much time has passed, which is to say $h = t_0 + h - t_0 = \Delta t_0$:

$$\frac{\Delta x(t_0)}{\Delta t_0}$$

This value is the **mean velocity** from instant t_0 to instant $t_0 + h$. However, the mean velocity is a value that only gives a general idea of the speed on some period of time. In this duration, the **instant velocity** (actual speed) can vary a lot and the mean velocity would then be far off to these values.

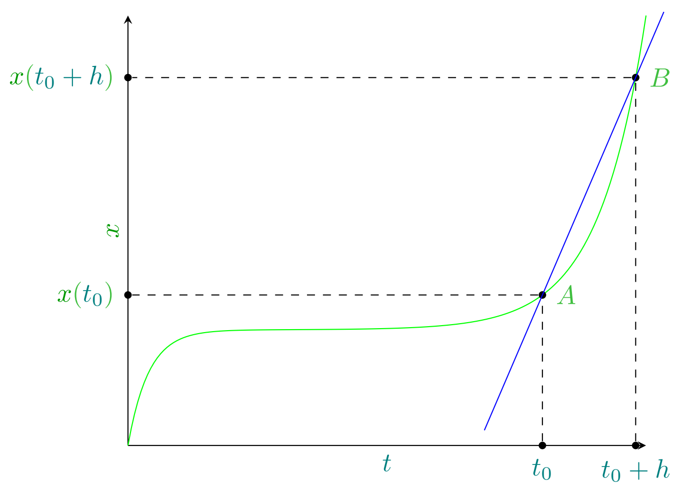


Figure 1.3: We add a point B to the previous graph at time $t_0 + h$. The mean velocity from A to B can be thought as the slope of the blue line (AB).

Since we expect the speed to not change a lot on short periods of time, a natural solution is to consider the mean velocity over shorter durations.

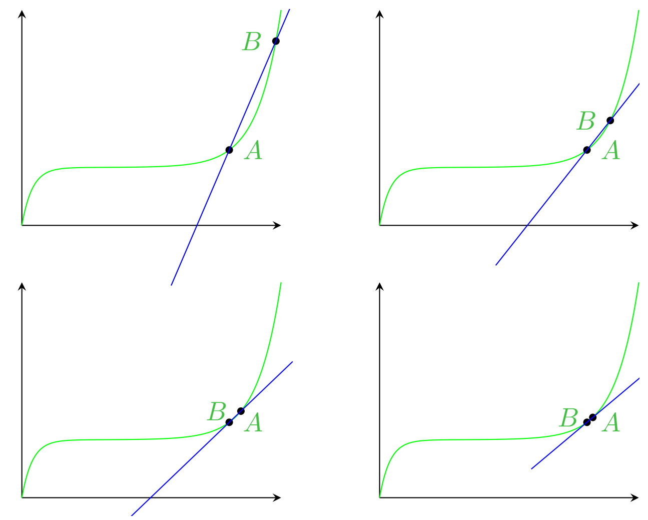


Figure 1.4: The closer to A we pick B, the best the blue line matches the curve at A.

So, as we pick shorter and shorter durations h, the value Δt_0 becomes smaller, but so does Δx. Often, we will notice that the mean velocity seems to **converge** (becomes closer and closer) to a particular value. Instead of continuing to choose smaller and smaller values of h, we will pick this values and call it the **limit** of $\frac{\Delta x(t_0)}{\Delta t_0}$ as h tends to 0 (becomes smaller and smaller). Or, for short:

$$\lim_{h \to 0} \frac{\Delta x(t_0)}{\Delta t_0}$$

Such a limit is called the **derivative** of x at t_0. We have a shorter way to note this:

$$\frac{\mathrm{d}x(t_0)}{\mathrm{d}t}$$

Here, the derivative of x at t_0 corresponds to the mean velocity over an infinitely small period, that is, the instant velocity.

Finally, we can do this for any value of t_0. Thus, we have a new function that let us evaluate the velocity at any t_0:

$$x' = \frac{\mathrm{d}x}{\mathrm{d}t}$$

When the derivation is done with respect to time (i.e. $\frac{\cdot}{t}$), we can simply use the dot notation: $\dot{x}$.

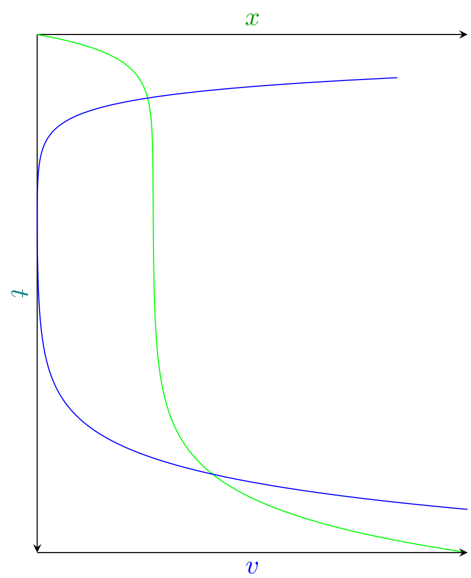


Figure 1.5: This graph features both position and velocity. Since they do not represent the same type of measurement, they each use a different axis and comparing the relative positions of their curves is meaningless. However, we can see that, as the position stabilizes in the middle, the velocity decreases; in the end the object moves again, faster and faster.

1.3.2 Second derivative

The velocity is the derivative of the position. As a function, it can itself fluctuate and we can be interested in these variations. The derivative of the velocity is the **acceleration**: $a = \dot{v}$.

A shorter way of saying that the acceleration is the derivative of *the derivative of the position*, is to say that the acceleration is the second derivative of the position: $a = \ddot{x}$.

1.3.3 Formal derivation

We now have a way to compute the derivative of a function at a given point. However, it is not accurate: while we do get a better approximation by taking a smaller value for h, the result is still an approximation and can sometimes stay far off.

Instead, we can look at the expressions to determine the exact value for the limit. For instance, let us consider the function $f(x) = 12x$ and let us search for the derivative of f at some x, i.e. $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$. First:

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{12(x+h) - 12x}{h} = \frac{12\cancel{h}}{\cancel{h}} = 12$$

We now look at the value $\frac{\Delta f(x)}{\Delta x}$ as Δx gets small; in this case, it happens to always be 12, and does not depend on Δx. Thus, however small Δx, the value is 12, and:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{h \to 0} \frac{\Delta f(x)}{\Delta x} = 12$$

That way, we know the exact value of derivative of f in any point. Let us take a second example with $g(x) = 7x^2$:

$$\frac{\Delta g(x)}{\Delta x} = \frac{7(x+h)^2 - 7x^2}{h} = \frac{7(x^2 + 2xh + h^2) - 7x^2}{h} = \frac{7x\cancel{h} + h^2}{\cancel{h}} = 7x + h$$

Here, the expression does depend on h. However, the smaller h gets, the less influence it has on the sum: the value is becoming closer and closer to $7x$. Thus: $\frac{\mathrm{d}g(x)}{\mathrm{d}x} = 7x$.

1.3.4 Derivation rules

Now, what if we want to compute the derivative of $h(x) = 12x + 7x^2$? Of course, we could go through the same step as in the previous part. However keeping the same definitions of f and g, we can notice that $h = f + g$. It means that $h(x) = f(x) + g(x)$ for all x's.

It turns out that it can be shown that $\frac{\mathrm{d}}{\mathrm{d}x}(f + g) = \frac{\mathrm{d}}{\mathrm{d}x}f + \frac{\mathrm{d}}{\mathrm{d}x}g$ for any functions f and g. Using this rule and knowing the derivative of f and g, we can derive:

$$\frac{\mathrm{d}}{\mathrm{d}x}h(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f+g)(x) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x) = 12 + 7x$$

There are a few derivation rules that can help us determine the derivative of complex functions easily.

$$\begin{aligned}
(\alpha f)' &= \alpha f' \\
(f+g)' &= f' + g' \\
(f \times g)' &= f' \times g' + g' \times f \\
\left(\frac{f}{g}\right)' &= \frac{f'g - g'f}{g^2} \\
(f(g(x)))' &= g'(x) \times f'(g(x))
\end{aligned}$$

(a) derivation rules

$$\begin{aligned}
(x^n)' &= nx^{n-1} \\
(e^x)' &= e^x \\
(\ln x)' &= \frac{1}{x} \\
(\cos x)' &= -\sin x \\
(\sin x)' &= \cos x
\end{aligned}$$

(b) common derivatives

1.4 Integrals

1.4.1 Definition

Now, let us consider the reverse situation: we know the velocity of the car v at any given instant and we would like to know where it was at an arbitrary instant t_0. In other words, we know the derivative of the position $\dot{x} = v$ and we intend to get the position x back.

The first thing to notice is that the velocity only informs us on *relative* motion: two cars can have the same velocity through time while being in different positions. In the example below, the positions of two cars with the same speed are shown; since the red car starts ahead, it stays ahead.

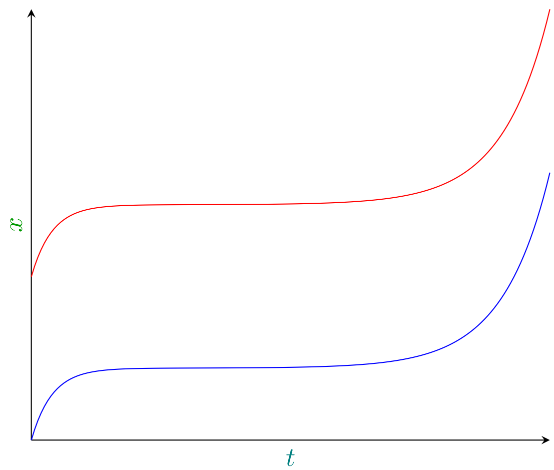


Figure 1.7: The red car moves like the blue car does, but starts ahead.

This means that we will need additional information to know where to start. Here, we will assume the car start at $x(0\ s) = 0\ m$.

As a first approximation, we could pretend the velocity is constant, and always equal to $v(t_0)$. That would make the position trivial to compute: $x(t_0) \simeq v(t_0) \times t_0$.

Now, given a constant velocity v_0 and a delay t_0, we know how to compute the distance x_0 as $x_0 = v_0 \times t_0$. In this situation however, the velocity changes over the time interval from $0\ s$ to t_0.

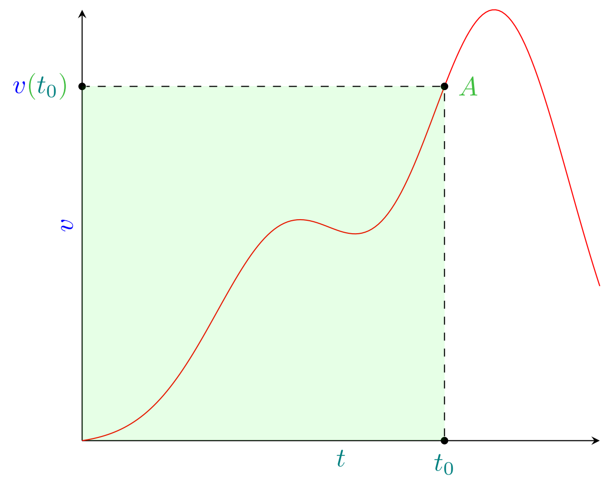


Figure 1.8: The graph clearly shows that the velocity may not be close to $v(t_0)$ (horizontal dotted line): this is a very rough first approximation.

Note that $v_0 \times t_0$ is also the area of a rectangle of height v_0 and width t_0. This maps to the green area on the graph.

For a better approximation, we will simply split this in several parts of width h.

For instance, we can assume that, from time $t = 0\ s$ to time $t = h$, velocity is constant and equal to $v(h)$. Then, the distance traveled on this duration is simply $v(h) \times h$; then, from h to $2h$, the car further travel $v(2h) \times h$. Thus, from time $t = 0\ s$ to time $t = 2h$, the car traveled roughly $v(h) \times h + v(2h) \times h$.

When we consider more steps, we will want to avoid writing the whole sum. Instead, we can use the Σ-notation ("sigma", Greek equivalent of s) to denote a sum:

$$\begin{aligned}\sum_{i=1}^{n} v(ih) \times h =& v(1h) \times h \\ &+ v(2h) \times h \\ &+ v(3h) \times h \\ &+ \dots \\ &+ v(nh) \times h\end{aligned}$$

In other words, $\sum_{i=1}^{n} v(ih) \times h$ means "sum the expression $v(ih) \times h$ where i takes each of the integer values from 1 to n".

Here, we will want to have $h \times n = t_0$ so that we can retrieve the distance traveled from time $t = 0\ s$ to time $t = t_0$.

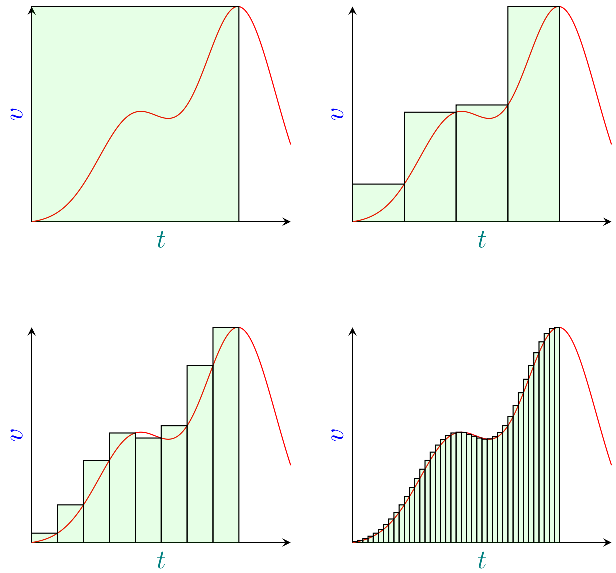


Figure 1.9: By assuming the velocity is constant on smaller and smaller time interval, our aproximation becomes more precise.

? Notice that the value we are looking for is the sum of the surface areas of the green rectangles. It turns out that this tends to match the value of the surface area under the curve.

As for derivation, when h tends to zero, our rough approximation will become more precise. Since we stil want $h \times n = t_0$ we will instead make n grow instead, and set h to $\frac{t_0}{n}$:

$$\sum_{i=1}^{n} v\left(i\frac{t_0}{n}\right) \times \frac{t_0}{n}$$

As n tends to infinity (grows larger and larger), we expect the sum to converge (come closer and closer) to some fixed value. Again, there is a short notation for this:

$$\int_{0\ s}^{t_0} v(t)\mathrm{d}t = \lim_{n\to\infty} \sum_{i=1}^{n} v\left(i\frac{t_0}{n}\right) \times \frac{t_0}{n}$$

An ever shorter notation when there is no ambiguity is simply:

$$\int_{0\ s}^{t_0} v$$

Notations with $\mathrm{d}x$ and $\mathrm{d}t$ make it easy to reason with derivatives and integrals. Since v is the derivative of x:

$$\int_{t_1}^{t_2} v = \int_{t_1}^{t_2} \frac{\mathrm{d}x}{\cancel{\mathrm{d}t}} \cancel{\mathrm{d}t} = \sum_{t_1}^{t_2} \mathrm{d}x = x(t_2) - x(t_1)$$

This highlights the fact that integrals are just summing up all the variations between two points (here, from $t = t_1$ to $t = t_2$). In particular, to know the exact value of $x(t_2)$, we need to know the initial value $x(t_1)$.

There is also a shorter notation for the difference of a value between two points:

$$x(t_2) - x(t_1) = [x]_{t_1}^{t_2}$$

1.4.2 Illustration

For example, if $v(t) = t \times 2\ m/s^2$, then:

$$\begin{aligned} x(30\ s) - x(0\ s) &= \int_0^{30\ s} t \times 2\ m/s^2 t\mathrm{d}t \\ &= \left(\int_0^{30\ s} 2t\mathrm{d}t\right) m/s^2 \\ &= [t^2]_0^{30\ s} m/s^2 \\ &= (900\ s^2 - 0\ s^2) m/s^2 \\ &= 900\ m \end{aligned}$$

In particular, with we are given the additional information that C did start at O, i.e. $x(0) = 0$, then:

$$x(30\ s) = 900\ m$$

This means that an object starting at rest getting a constant push of $2\ m/s^2$ will travel $900\ m$ in $30\ s$.

1.4.3 Antiderivative

More generally, in the previous example, we could write:

$$x(t) - x(0) = \int_0^{30\ s} 2\ m/s^2 \times t\mathrm{d}t = t^2 m/s^2$$

Another way to write it is:

$$x(t) = t^2 m/s^2 + C$$

where $C = x(0)$ is a value independent of t which depends on the **initial conditions** (e.g. the position of the car at the initial instant). Each of the possible expression of x (depending on C) is a primitive of $\dot{x}$.

1.4.4 Geometric integrals

There should be more explanations here, but these formulas shows how we compute the areas and volumes of common shapes.

Circle circumference:

$$\begin{aligned} d &= \oint_C \mathrm{d}s \\ &= \int_0^{2\pi} R\mathrm{d}\theta \\ &= R\,[\theta]_0^{2\pi} \\ &= 2\pi R \end{aligned}$$

Disk area:

$$\begin{aligned} A &= \iint_D \mathrm{d}A \\ &= \int_{r=0}^{R}\int_{\theta=0}^{2\pi} \mathrm{d}r \times r\mathrm{d}\theta \\ &= 2\pi \int_0^R r\mathrm{d}r \\ &= 2\pi \left[\frac{1}{2}r^2\right]_0^R \\ &= \pi R^2 \end{aligned}$$

Sphere area:

$$\begin{aligned} A &= \oiint_S \mathrm{d}A \\ &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (R\sin\phi\mathrm{d}\theta)(R\mathrm{d}\phi) \\ &= 2\pi R^2 \int_0^{\pi} \sin\phi\mathrm{d}\phi \\ &= 2\pi R^2\,[-\cos\phi]_0^{\pi} \\ &= 4\pi R^2 \end{aligned}$$

Sphere enclosed volume:

$$
\begin{aligned}
V &= \iiint_B \mathrm{d}V \\
&= \int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\sin\phi\,\mathrm{d}\theta)(r\,\mathrm{d}\phi)\,\mathrm{d}r \\
&= 2\pi\int_{r=0}^{R}\int_{\theta=0}^{\pi}\sin\phi\,\mathrm{d}\phi\, r^2\,\mathrm{d}r \\
&= 2\pi\left[\frac{1}{3}r^3\right]_0^R[-\cos\phi]_0^{\pi} \\
&= \frac{4}{3}\pi R^3
\end{aligned}
$$

1.5 Differential equations

1.5.1 Definition

A differential equation is an equation whose unknown is a function and involving a derivative of this function. For example:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 3x^2$$

This is a differential equation and we already know how to solve it (find the expression of f). Here, $f(x) = x^3 + C$ for any constant value C (there are several possible solutions).

1.5.2 Exponential

The exponential function is defined as $f(x) = e^x$ where e is a mathematical constant whose value is about 2.718. It was picked so that:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f$$

In other words:

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$$

? If we define $f(x) = e^{g(x)}$ instead, derivation rules gives us:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = g'(x)e^{g(x)}$$

so that $\frac{\mathrm{d}f}{\mathrm{d}x} = g'f$.

1.5.3 First order

Now, consider the following equation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f$$

We already know that $f(x) = e^x$ is a solution; however, so is $f(x) = 2e^x$. Actually, the set of solutions to this equation is the functions $f(x) = Ce^x$ where C is any constant value.

The remark we made before tell us how to solve a differential equation of the form:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = hf$$

where h is also a function. We just need to find g such that $g' = h$, i.e. the solutions are:

$$f(x) = Ce^{\int_0^x h(x)\mathrm{d}x}$$


MINMUS

Chapter 2

Mechanics

> Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.
>
> – Isaac Newton, discoverer of the laws of motion and gravitation

Mechanics is the study of movement. In this chapter, we will describe how physicists study the movement of physical objects using mathematical laws. With this knowledge, we will then see how this is relevant to rocket science.

2.1 Referential

2.1.1 Origin

The first thing we want to do is to find a way to describe **where** objects are. For this, we will elect an object (an actual physical object or just an arbitrary virtual point in space) to be fixed, and consider all movements relative to this **origin**.

2.1.2 Dimensions

If all the objects are on a same line (for instance a race track), then we can describe the position of each object with a single measure: the distance from the origin. Since a single measure is sufficient, a line is said to have one dimension.

If all the objects are on a same surface (for instance, on a checkboard), then we need two numbers to fully describe the position of any of these objects. For example, we may arrange items on a grid a tell their positions as the row and column numbers; we can locate a location on the surface of the Earth using longitude and latitude. Since two measures are needed, a surface is said to have two dimensions.

There are many ways to choose how to describe the position of an object. For instance, rather than using the row and column number, we could use the distance from the center and the angle formed with a fixed line. When we say "there are two dimensions", the "two dimensions" do not refer to any particular measures, but to the necessity of two different ones. There are many choices of two measures to describe two dimensions.

Finally, we can consider an arbitrary location in space, which needs three dimensions. For example, if we add altitude to longitude and latitude, we can easily locate any object.

A referential is the object we use as a landmark (origin) to keep track of interesting points. A system of coordinates if the kind of data you use to store the position of these points relatively to the origin.

The most common set of measures used in two dimensions are the **Cartesian coordinate** and the **polar coordinate** systems. The Cartesian coordinate system extend to three dimensions naturally; there are two ways to extend the polar coordinates system to a third dimension: the **cylindrical coordinate** and the **spherical coordinate** systems.

2.1.3 Cartesian coordinates

Cartesian coordinates are the most common. We basically choose two (or three, in space) directions, and the measures describes how much you have to go each way to get from the origin to the point.

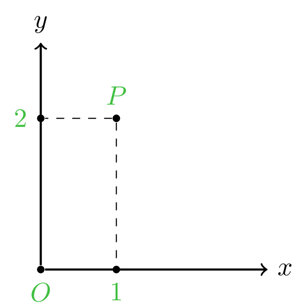


Figure 2.1: P is at coordinates $(1, 2)$ in this referential: to go from origin O to O, we have to go along direction $\vec{x}$ for 1 unit, and along direction $\vec{y}$ for 2 units

The directions used in Cartesian coordinates (the axes) are usually perpendicular and conventionally named x and y, (and z)).

Derivatives

A nice property of the Cartesian coordinate system is that, if we are interested in the variation of some point P with Cartesian coordinates (x, y), we can simply consider the **vector** (x, y).

While a point denotes an offset from the origin, a vector denotes an offset from an unspecified point.

2.1.4 Polar coordinates

Polar coordinates work in two dimensions. One value is simply the distance to the origin while a second is the angle of P with a fixed porientation.

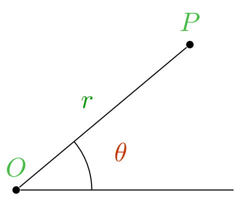


Figure 2.2: The polar coordinates of P are (r, θ)

Derivative

Consider a point P given in polar coordinates (θ, r). We can use the corresponding Cartesian coordinates: $P = (r\cos(\theta), r\sin(\theta))$. Then we can derive on each Cartesian coordinate using the derivation rules for function composition.

If we define $\hat{r} == (\cos\theta, \sin\theta)$ and $\hat{\theta} = (-\sin\theta, \cos\theta))$, then:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{r} = (-\dot{\theta}\sin\theta, \dot{\theta}\cos\theta) = \dot{\theta}\hat{\theta} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\theta} = (-\dot{\theta}\cos\theta, -\dot{\theta}\sin\theta) = -\dot{\theta}\hat{r} \end{cases}$$

Now, P is at an offset $\vec{r}$ to the origin, where:

$$\vec{r} = (r\cos(\theta), r\sin(\theta)) = r\hat{r}$$

thus:

$$\dot{\vec{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(r\hat{r}\right) = \hat{r}\frac{\mathrm{d}}{\mathrm{d}t}r + r\frac{\mathrm{d}}{\mathrm{d}t}\hat{r} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

By deriving again, we get the acceleration:

$$\ddot{\vec{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

2.1.5 Spherical coordinates

Spherical coordinates are simply the generalization of polar coordinates to three dimensions. To the usual coordinates r and θ, we add a third one, ϕ, which is the angle with the polar plane:

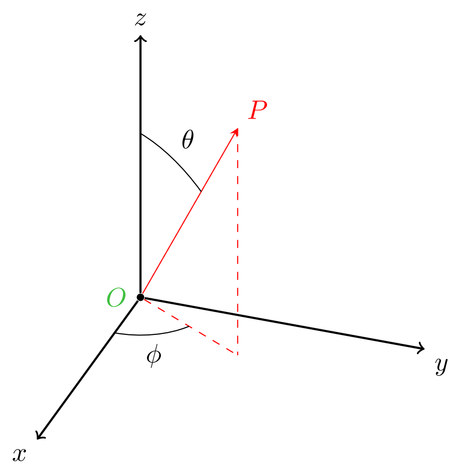


Figure 2.3: The spherical coordinates of P are (r, θ, ϕ)

2.2 Newtonian mechanics

2.2.1 Center of mass

For the sake of simplicity, we will pretend that objects are simple points (e.g. P) associated to their mass (e.g. m); such points (e.g. (P, m)) are called mass-points. For a given object, the point we will use is called the center of mass (CoM). It is easy to find for regular and uniform objects (e.g. a sphere).

We will usually refer to the position P by using the vector $\vec{r} = \overrightarrow{OP}$. The velocity is simply the derivative of the position: $\vec{v} = \dot{\vec{r}}$; the acceleration is the derivative of the velocity: $a = \dot{\vec{v}} = \ddot{\vec{r}}$.

2.2.2 Newton's second law

Consider a mass-point (P, m) subjected to forces $\vec{F}_1, \vec{F}_2, \ldots$. We need not consider every force independentely and can sum them as the **resultant force** $\vec{F} = \sum \vec{F}_i$.

Then, according to Newton's second law:

$$\vec{a} = \frac{1}{m} \vec{F}$$

This equation means several things. First, it means that an object on which is exerted a non-zero resultant force will be accelerated (acceleration encompasses the slowing down of an object). Second, the more mass it has, the less it will accelerate; since this is perceived as a resistance to movement, we call this inertia.

When $\vec{F} = 0$, the acceleration is null as well. Thus, the speed is constant; this is Newton's first law (chapter quote).

2.2.3 Gravitation

The gravitational force is one of the four fundamental forces that physicist have identified, along with the electromagnetic, weak and strong forces.

Gravitation says that any two objects with a mass attract each other, at any distance. Since it is extremely feeble, we do not observe it within common objects. It becomes perceptible when huge masses are grouped together; for instance, we observe an apple falling because of the vast mass of the Earth effecting the apple.

The amount of force exerted by an object of mass M over an object of mass m is:

$$F = \mathcal{G}\frac{Mm}{r^2}$$

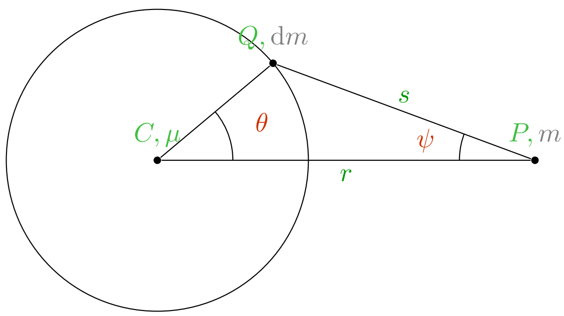
where r is the distance between the two objects.

The object of mass m attracts the object of M with the exact same mount of force. The reason the Earth does not seem to move towards a falling apple is because of inertia (see Newton's second law).

2.3 Shell theorem

2.3.1 Problem

We are interested in knowing the gravitation exerted by a celestial body C (i.e. the gravity). We will assume the celestial body is a ball of radius R and uniform density μ and consider a mass-point (m, P) at distance r from C:



2.3.2 Elementary force

The usual way to solve a large problem is to split it in smaller problems and solve those instead. Here, to find the total force exerted, we will split the spherical body in many mass points $(Q, \mathrm{d}m)$.

The force exerted by Q over P is oriented along $\vec{QP}$ and has magnitude:

$$F = \mathrm{d}\vec{g}m = \mathcal{G}\frac{m\mathrm{d}m}{PQ^2}$$

Another way to write this is:

$$\mathrm{d}\vec{g} = \mathcal{G}\frac{\mu\mathrm{d}V}{s^2}$$

where s is just PQ

2.3.3 Symmetry

Now, to simplify things, we can notice the axial symmetry of the whole situation around (CP). This means that the resultant force $\vec{g}$ can only be along the axis (CP): no other could be justified without breaking the symmetry.

Thus, we can only choose to only consider the effect of Q along (CP); the others effect will cancel out. The effect of gravitation from Q exerted on P along (CP) is:

$$\mathrm{d}\vec{g} \cdot \frac{\vec{PC}}{PC} = \mathcal{G}\frac{\mu\mathrm{d}V}{s^2} \times \cos\psi$$

where ψ is the angle $C\hat{P}Q$.

2.3.4 Integrating

Now, it is only a matter of integrating. We first extend the elementary volume $\mathrm{d}V$ in spherical coordinates as $(\rho\sin\psi\mathrm{d}\theta)(\rho\mathrm{d}\psi)$. Then we just need to do a substitution and to cancel out terms.

$$
\begin{aligned}
g &= \iiint_S \mathrm{d}\vec{g} \cdot \frac{\overrightarrow{PC}}{PC} \\
&= \iiint \frac{\mathcal{G}\mu \mathrm{d}V}{s^2} \cos\psi \\
&= \mathcal{G}\mu \int_{\rho=\rho_-}^{\rho^+} \int_{\psi=0}^{\alpha} \int_{\theta=0}^{2\pi} \frac{\cos\phi}{\rho^2} \times (\mathrm{d}\rho)(\rho \sin\psi \mathrm{d}\theta)(\rho \mathrm{d}\psi) \\
&= 2\pi\mathcal{G}\mu \int_{\rho=\rho_-}^{\rho^+} \int_{\psi=0}^{\alpha} \cos\psi \sin\psi \mathrm{d}\psi \mathrm{d}\rho \\
&= 2\pi\mathcal{G}\mu \int_{\psi=0}^{\alpha} 2\sqrt{R^2 - r^2\sin^2\psi}\cos\psi \sin\psi \mathrm{d}\psi \mathrm{d}\rho && (2.1) \\
&= 4\pi\mathcal{G}\mu \int_{u=R}^{\cancel{0}} u\sqrt{\cancel{1} - \frac{\cancel{R^2 - u^2}}{r^2}} \frac{\cancel{\sqrt{R^2-u^2}}}{r} \frac{\cancel{-}u}{\cancel{\sqrt{r^2 - R^2}} + u^2 \cancel{\sqrt{R^2-u^2}}} \mathrm{d}u && (2.2) \\
&= 4\pi\mathcal{G}\mu \frac{1}{r^2} \int_0^R u^2 \mathrm{d}u \\
&= \mathcal{G}\mu \underbrace{\frac{4}{3}\pi R^3}_{=V} \times \frac{1}{r^2}
\end{aligned}
$$

2.3.5 Result

In the end, $g = \mathcal{G}\frac{M}{s^2}$. It means that, the resultant gravitation force exerted by a planet is the same as the mass-point in the center with the same mass, even when close to it.

This will make all considerations involving the gravitation of a planet relatively easy, as long as we assume it is spherical and uniform.

2.3.6 Details on the substitution

To get from (2.1) to (2.2), we substitute $u = \sqrt{R^2 - r^2\sin^2\psi}$ for ψ. This means that $\psi = \arcsin\frac{\sqrt{r^2-u^2}}{r}$ and subsequently:

$$
\mathrm{d}\psi = \frac{1}{\sqrt{1 - \frac{R^2-u^2}{r^2}}} \times \frac{-2u\mathrm{d}u}{2r\sqrt{R^2-u^2}} = -\frac{u}{\sqrt{r^2 - R^2 + u^2}\sqrt{R^2-u^2}}\mathrm{d}u
$$

We then use the relations $\sin(\arcsin x) = x$ and $\cos(\arcsin x) = \sqrt{1-x^2}$ for $0 \leq x \leq \frac{\pi}{2}$.

2.4 Sphere of influence

2.4.1 Patched conics

In reality, an object travelling through space is influenced by all moons, planets and stars of the universe. Even when restricting to the closest moon, closest planet and closest star, the combined influence of several bodies is hard to take into account simultaneously.

A simple approximation is to only consider the body with the most influence. The region where a celestial body has the most influence is called the **sphere of influence**. When leaving the sphere of influence of one body, we transition to that of another. The trajectory is then made of two parts (two conics), hence the name **patched conics** approximation.

2.4.2 Influences

We will consider two mass-points (P_1, M_1) and (P_2, M_2) and look at their influences on a mass-point (P, m) between them.

The intensity of forces F_1 and F_2 respectively exerted by P_1 and P_2 are:

$$F_1 = \mathcal{G}\frac{M_1 m}{P_1P^2} \text{ and } F_2 = \mathcal{G}\frac{M_2 m}{P_2P^2}$$

For short, we define $r = P_1P$ and $R = P_1P_2$. Thus, $P_2P = P_1P_2 - P_1P = R - r$. We also define $\mu_1 = \mathcal{G}M_1$ and $\mu_2 = \mathcal{G}M_2$ (the **gravitational parameters**). The accelerations due to each are:

$$g_1 = \frac{\mu_1}{r^2} \text{ and } g_2 = \frac{\mu_2}{(R-r)^2}$$

The acceleration exerted by P_1 over P_2, and by P_2 over P_1 are respectively:

$$g_{1,2} = \frac{\mu_1}{R^2} \text{ and } g_{2,1} = \frac{\mu_2}{R^2}$$

2.4.3 Perturbations

Now, if we look at the acceleration of P **relatively** to P_1, it is **perturbed** by the fact that P_2 does not exert the same acceleration over P and over P_1. The absolute perturbation is $g_2 - g_{2,1}$ (the difference in acceleration from P_2).

In practice, this value only makes sense when compared to the **main** acceleration g_1. Here, we will assume that $r \ll R$; it makes sense when P_1 is an object significantly smaller than P_2. We define the relative perturbation Q_1 as:

$$Q_1 = \frac{g_2 - g_{2,1}}{g_1} = \frac{\frac{\mu_2}{(R-r)^2} - \frac{\mu_2}{R^2}}{\frac{\mu_1}{r^2}} = \frac{r^2}{R^2}\frac{\mu_2}{\mu_1}\left(\frac{1}{\left(1-\frac{r}{R}\right)^2} - 1\right)$$

Now, it can be shown that, when x is small, $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots \simeq 1 + 2x$. With $x = \frac{r}{R}$ we get that:

$$Q_1 = \frac{r^2}{R^2}\frac{\mu_2}{\mu_1} \times 2\frac{r}{R} = 2\frac{r^3}{R^3}\frac{\mu_2}{\mu_1}$$

Similarly, the relative perturbation due to P_1 when looking at the acceleration of P relatively to P_2 is:

$$Q_2 = \frac{g_1 - g_{1,2}}{g_2} = \frac{\frac{\mu_1}{r^2} \cancel{- \frac{\mu_1}{R^2}}}{\frac{\mu_2}{(R \cancel{- r})^2}} = \frac{\mu_1}{\mu_2}\frac{R^2}{r^2}$$

Again, we use the fact that $x \ll X$ to simplify the expression.

2.4.4 Radius of influence

Now, we are interested in the point P between P_1 and P_2 where both relative perturbations are of the same magnitude. We thus solve:

$$\begin{aligned} Q_1 = Q_2 \Leftrightarrow 2\frac{r^3}{R^3}\frac{\mu_2}{\mu_1} &= \frac{\mu_1}{\mu_2}\frac{R^2}{r^2} \\ \Leftrightarrow r &= \frac{1}{2^{\frac{1}{5}}} R \left(\frac{\mu_1}{\mu_2}\right)^{\frac{2}{5}} = 2^{-\frac{1}{5}} R \left(\frac{M_1}{M_2}\right)^{\frac{2}{5}} \end{aligned}$$

The $2^{-\frac{1}{5}}$ factor seems to be forgotten often. Since its value is about 0.87, it is still the right order of magnitude.

2.5 Thrust

2.5.1 Propulsion

A car can push along the road thanks to friction; a plane can generate lift from air and speed; a balloon can use buoyancy in the atmosphere. However, a rocket needs to be able to operate in the vacuum of outer space, which implies no physical object to push against.

Articulating an object into space will not move its center of mass. It means that a rocket can only gain speed by interacting with the exterior. Thus, the only way a rocket can accelerate is by throwing parts of its mass out. This is exactly what their exhaust do: it ejects lots of mass (propellant) down, so that the rocket will lift up.

Technically, outer space is not empty: light from the Sun is theoretically enough for the low-thrust propulsion of **solar sails**.

2.5.2 Conserved momentum

We consider a **r**ocket with center R of mass m. From t to $t + \mathrm{d}t$, it ejects $\mathrm{d}m$ of its **p**ropellant (P) with relative (**exhaust**) velocity v_e.

Let us notate G the center of mass of the system $\{R, P\}$. Since the system did not interact with its environment, the speed of G remains unchanged by the ejection. However, since P is moving downwards, it means the rocket have gained speed.

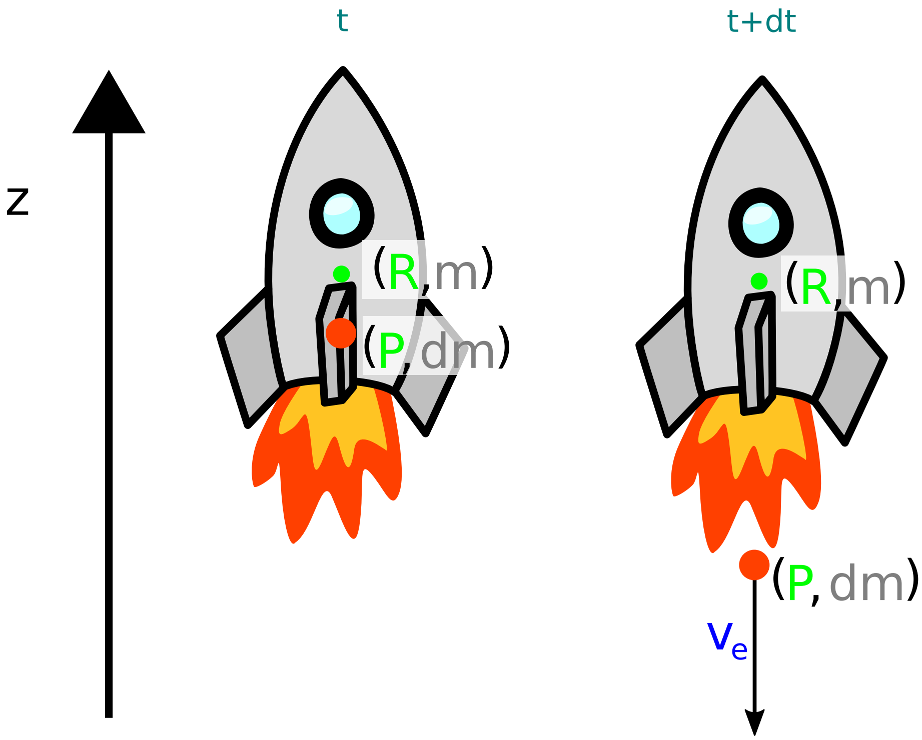


Figure 2.4: Before time t, R and P are moving together. Then, P is ejected downwards at speed v_e. Since the **momentum** of the pair $\{R, P\}$ has not changed, it means that the rocket has gained some speed upwards.

2.5.3 Exerted force

We are interested in the changes of speed $\mathrm{d}v_R$ and $\mathrm{d}v_P$ from t to $t + dt$. If we do not take into account external forces, then $\mathrm{d}v_g = 0$. Thus:

$$\begin{aligned}
0 &= \mathrm{d}v_G \\
&= \mathrm{d}v_R m + \mathrm{d}v_P \mathrm{d}m \\
&= \mathrm{d}v_R(m + \mathrm{d}m) + v_e \mathrm{d}m \qquad \text{because } v_P = v_R - v_e
\end{aligned}$$

Dividing by dt and with $m + \mathrm{d}m \simeq m$, it follows that:

$$m\frac{\mathrm{d}v_r}{\mathrm{d}t} = -v_e\frac{\mathrm{d}m}{\mathrm{d}t}$$

We finally get the expression of the thrusting force:

$$F_t = ma_R = -v_e\dot{m} \tag{2.3}$$

2.5.4 Specific impulse

In KSP, the engines are defined by their maximum thrust F_t and their I_{sp} ("atmosphereCurve" in configuration files). The **sp**ecific **i**mpulse is just the force exerted per unit (in weight) of fuel used, i.e. $I_{\mathrm{sp}} = \frac{F_t}{\dot{m}g}$. Thus:

$$\dot{m} = \frac{F_t}{I_{\mathrm{sp}} \times g} \text{ and } \quad v_e = -I_{\mathrm{sp}} \times g$$

2.5.5 Tsiolkovsky rocket equation

We can now integrate equation (2.3) to get the velocity change of a rocket over a long period of time:

$$\begin{aligned}
\Delta v &= \int_0^t \dot{v_R}\mathrm{d}t \\
&= \int_0^t -v_e\frac{\dot{m}}{m}\mathrm{d}t \\
&= [-v_e \ln(m)]_0^t \\
&= v_e \ln\frac{m(0)}{m(t)} \qquad (2.4)
\end{aligned}$$

Conversely, we can compute the amount of propellant Δm to eject to bring a mass m to a given speed:

$$\Delta m = m\left(1 - e^{-\frac{\Delta v}{v_e}}\right)$$


GILLY

Chapter 3

Orbits

> “There is a force in the earth which causes the moon to move.”
>
> – Johannes Kepler, discoverer of the laws of orbits

Before launching a rocket, we need to know where we are going. Up is only part of the answer.

3.1 Orbital trajectory

3.1.1 Concept

First, the obligatory quote when explaing how orbiting works:

> “The knack [of flying] lies in learning how to throw yourself at the ground and miss.”
>
> – Double Adams, author of *The Hitchhiker’s Guide to the Galaxy*

The part of throwing oneself at the ground is easy:

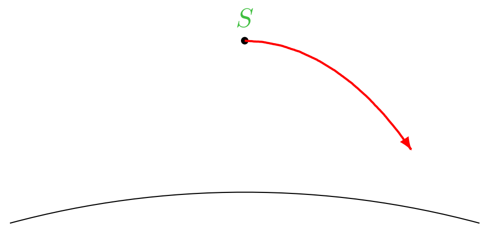


Figure 3.1: Most of the time, stuff we throw into the air falls down.

On the figure above, notice that, as we go right, the ground gets lower and lower due to the **curvature** of the planet. Can we go right fast enough so that the ground will lower as fast as we fall? Well, yes:

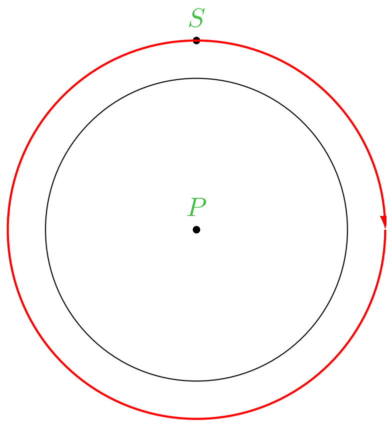


Figure 3.2: If we go fast enough, we will keep missing the ground.

3.1.2 Central force

The point S is the **s**atellite whose trajectory we want to determine. The celestial objects it orbits is called the **primary**. Thanks to the shell theorem, we can assimilate the **p**rimary with a mass-point (P, M). Using Newton's Second Law, the acceleration due to the force of gravitation is:

$$\ddot{\vec{r}} = -\frac{\overbrace{\mathcal{G}M}^{\mu}}{r^2}\hat{r}$$

where $\vec{r} = \vec{PS}$ and $\hat{r} = \frac{\vec{PS}}{PS}$ is the direction (**unit vector**) from P to S.

? This already tells us that the acceleration, and hence the trajectory, does not depend on the mass of the satellite.

$$\begin{aligned}
\ddot{\vec{r}} &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(r\hat{r}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}
\end{aligned}$$

Hence the equation can be written as::

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = -\dfrac{\mu}{r^2} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \end{cases}$$

3.1.3 Kepler's second law

The second line can be rewritten as:

$$\begin{aligned} 0 &= 2\dot{r}\dot{\theta} + r\ddot{\theta} \\ &= \frac{1}{r}\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) \\ &= \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) \end{aligned}$$

This means that $h = r^2\dot{\theta}$ is constant.

3.1.4 Variable substitution

Now, the trick is to introduce a new variable $u = \frac{1}{r}$ and to consider its relation to θ. This means wil will replace r, $\dot{r}$ and $\ddot{r}$ with expressions involving $u(\theta)$. For short, we will use the following notation:

$$u' = \frac{\mathrm{d}u}{\mathrm{d}\theta} \text{ and } u'' = \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}$$

First, we have:

$$r = \frac{1}{u}$$

Then, we look at the expression of u':

$$u' = \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{r} = \frac{\mathrm{d}t}{\mathrm{d}\theta}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{r} = \frac{1}{\dot{\theta}} \times \left(-\frac{\dot{r}}{r^2}\right) = -\frac{\dot{r}}{h}$$

From this, we get that:

$$\dot{r} = -hu'$$

Similarly, we develop the expression of u'':

$$\begin{aligned} u'' &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{\dot{r}}{h}\right) \\ &= -\frac{1}{h}\frac{\mathrm{d}t}{\mathrm{d}\theta}\frac{\mathrm{d}}{\mathrm{d}t}\dot{r} \\ &= -\frac{1}{h}\frac{1}{\dot{\theta}}\ddot{r} \\ &= -\frac{\ddot{r}}{h\frac{h}{r^2}} && \text{because } h = r^2\dot{\theta} \\ &= -\frac{\ddot{r}}{h^2u^2} \end{aligned}$$

This let us know that:

$$\ddot{r} = -h^2u^2u''$$

3.1.5 Solving the equation

We can now rewrite the equation by using u, u' and u'':

$$\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= -\frac{\mu}{r^2} \\
-h^2u^2u'' - \frac{1}{u}(hu^2)^2 &= -\mu u^2 \\
-h^2\cancel{u^2}u'' - h^2u^{\cancel{3}} &= -\mu \cancel{u^2} \\
u'' + u &= \frac{\mu}{h^2}
\end{aligned}$$

It can be shown that the solutions for this equation are of the form:

$$u = \frac{\mu}{h^2} - A\cos(\theta - \theta_0)$$

for some values of A and θ_0. Thus:

$$r = \frac{1}{\frac{\mu}{h^2} - A\cos(\theta - \theta_0)}$$

Now, if we assume $\theta_0 = 0$ and set $l = \frac{h^2}{\mu}$ and $e = \frac{A}{p}$, then:

$$r = \frac{l}{1 - ecos(\theta)}$$

? e is called the **eccentricity** and l the **semi-latus rectum**

3.1.6 Shape

From this expression of r depending on θ, we can vizualize the possibles shapes for the trajectories of an objects only subjugated to gravitation.

First, we notice that the semi-latus rectum, l, only affects the scale: the larger l, the larger r, in all directions (all values of θ). It remains to look for the influence of the eccentricity, e.

First, notice that $\cos(\theta)$ is an **even** function. It means that $\cos(-\theta) = \cos(\theta)$, for all values of θ. It implies that $r(\theta)$ is also even: the trajectory is symmetric along the axis where $\theta = 0$ (also called the **major axis**).

Second, notice that $-\cos(\theta) = \cos(\theta + \pi)$. This means that reverting the value of e is the same as doing a 180 deg rotation. For this reason, we can restrict ourselves to non-negative values of e ($e \geq 0$).

Now, let us look at different cases:

If we set $e = 0$, we get rid of most of the expression and end up with $r(\theta) = l$. This means that the distance to the origin does not depend on the direction; in other words, the trajectory is a **circle**.

Because of the fraction, we need to avoid its denominator, $1 - e\cos(\theta)$ to be zero. Since $-1 \leq \cos(\theta) \leq 1$, choosing $0 < e < 1$ will ensure that $1 - e\cos(\theta) \neq 0$. That way, $r(\theta)$ is well defined in all directions; it turns out the shape is that of an **ellipse**.

On the other hand, if we choose $e \geq 1$, we will encouter a discontinuity: since $|\cos(\theta)|$ ranges from 0 to 1, there will be a values of θ where $1 - e\cos(\theta) \leq 0$, which means an infinite or negative value for $r(\theta)$. In practice this means that the object will never be in that direction: this is a **hyperbola** (or **parabola**, in the edge case where $e = 0$).

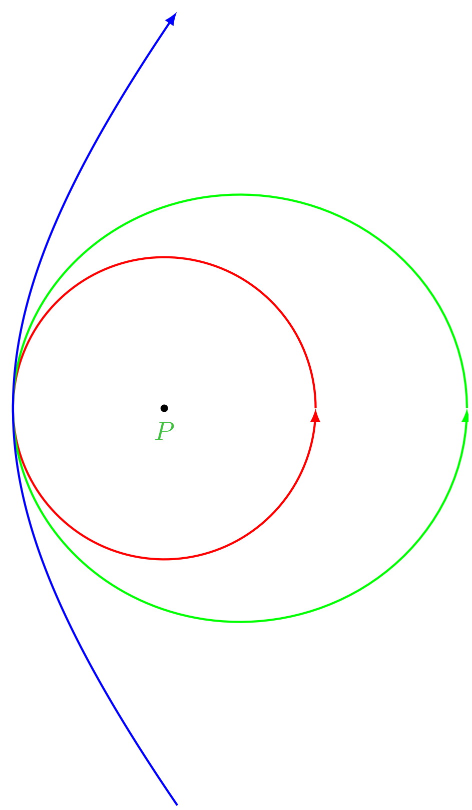


Figure 3.3: orbital trajectories arount primary P: circular orbit in red, elliptic orbit in green, and hyperbolic orbit in blue

? An object in free fall is always following an orbital trajectory around its primary. It is actually said to be **in orbit** when that trajectory does not

intersect the surface (nor goes to deep into the atmosphere).

When the orbit is closed (circular or elliptical), the oject is said to be **orbiting** its primary. When the orbit is open (hyperbolic or parabolic), the trajectory is said to be:

- a **capture orbit** (or trajectory) when moving towards the periapsis
- an **escape orbit** (or trajectory) when moving away from it

3.2 Characterisitics

3.2.1 Apses

We easily find the extremal value of r: the extremal values of $\cos(theta)$ are -1 and 1, which means that r varies between

- $r_{\text{per}} = \frac{l}{1+e}$ and $r_{\text{ap}} = \frac{l}{1-e}$ when $e < 1$
- $r_{\text{per}} = \frac{l}{1+e}$ and $+\infty$ when $e \geq 1$

The point where $r = r_{\text{per}}$ is called the **periapsis** (*peri-* is Greek for close), while the point where $r = r_{\text{ap}}$ is called the **apoapsis** (*apo-* is Greek for far).

?

While the periapsis is always defined, the apoapsis may not, since an infinite (or negative) altitude will never be reached.

When we want to be explicit about the object being orbited, we can use more specific suffixes that -apsis. For instance, the **perigee** is the periapsis of an object orbiting the Earth.

Table 3.1: Suffixes for various bodies

Body	Suffix
Galactic center	-galacticon
Star	-astron
Sun	-helion
Mercury	-hermion
Venus	-cytherion
Earth	-gee
Moon	-lune, -cynthion, -selene
Mars	-areion
Jupiter	-zene, -jove
Saturn	-krone, -saturnium
Uranus	-uranion
Neptune	-poseidon
Pluto	-hadion

? Various names come from Latin an Greek. For instance, Poseidon is the Greek god of the ocean, while Neptune is the Roman equivalent. The suffix *-gee* is related to Gaia, the Greek goddess of the Earth.

3.2.2 Angles

Astronomers use several ways to measure what are essentially angles.

The **true anomaly** is simply the angle θ we have been using in this chapter; it is also conventionally named f. In the case of an object in a non-circular orbit, this angle does not change uniformly. There are two reasons:

1. similar changes in position will result in smaller changes in true anomaly the further S is from P
2. the object is moving slower when it is far from P

The first issue is solved by measuring the angle from the center O, and projecting the trajectory on a regular circle. This new angle is named the **eccentric anomaly**, since it is measured *from* (*ex* in Latin) the *center*. It is usually notated E.

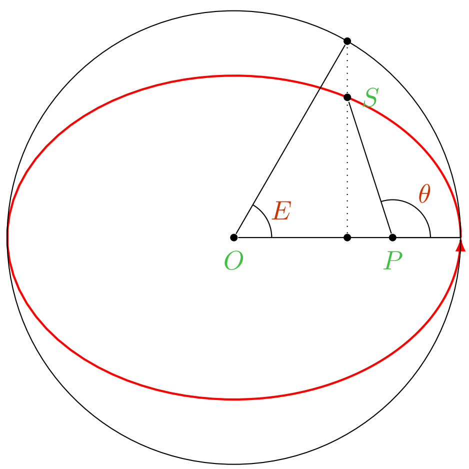


Figure 3.4: The orbital trajectory of S (in red) is projected to a circle (in black) and the angle measured from O is the eccentric anomaly E.

As for the second issue, Kepler's equation defines a new quantity M, the **mean anomaly**:

$$M = E - e\sin(E)$$

It can be shown that M grows linearly with time. It means that we can easily predict the mean anomaly of an object at time t as:

$$M(t) = M(0) + n \times t$$

where n is a constant value called the **mean motion**. By knowing the **mean anomaly at epoch** $M(0)$ of an object as well as its mean motion, we can retrieve its mean anomaly; we can then solve Kepler's equation to retrieve E, and then θ.

3.2.3 Orbital elements

The state of a mass-point is its position and velocity. In space, we need three values for each the position and the velocity, for a total of six parameters. The orbital elements are six such parameters that make it easy to also know the orbit of an object.

First, we have seen that the eccentricity, e, determines the shape of the orbit. Second, the semi-latus rectum determines the shape, but the periapsis r_{per} conveys the same information (it is always defined), and is more intuitive.

To tell where the object is on its orbit, we need an angle. The true anomaly is intuitive, but does not change regularly through time; for this reason, we will usually prefer the mean anomaly M.

We now need to rotate the orbit in its orbital plane. Since the periapsis is not necessarily on the x axis, we need an additional angle: the **argument of periapsis** ω.

Our orbit is now fully determined in two dimension; we just need to orient the orbital plane in space. For this, we will need to know the **inclnation** i, as well as where the inclination occurs.

The axis around which the orbit is inclined is named the **line of nodes**, and the two points its intersection with the the orbit are called the **ascending node** and the **descending node**. Their names corresponds to wether the object is going up (along the z axis) or down, when going through it. The line of nodes is simply determined by the **longitude of ascending node**, Ω.

To sum up, the six orbital elements are:

- periapsis, r_{per}
- eccentricity e
- argument of periapsis ω
- inclination i
- longitude of ascending node Ω
- mean anomaly M

3.3 *Vis viva* equation

The *vis viva* equation is a simple relation between speed and distance, which uses only one orbital characteristic.

The principle of energy conservation tells us that the total energy (kinetic and potential energies) of an isolated system is constant:

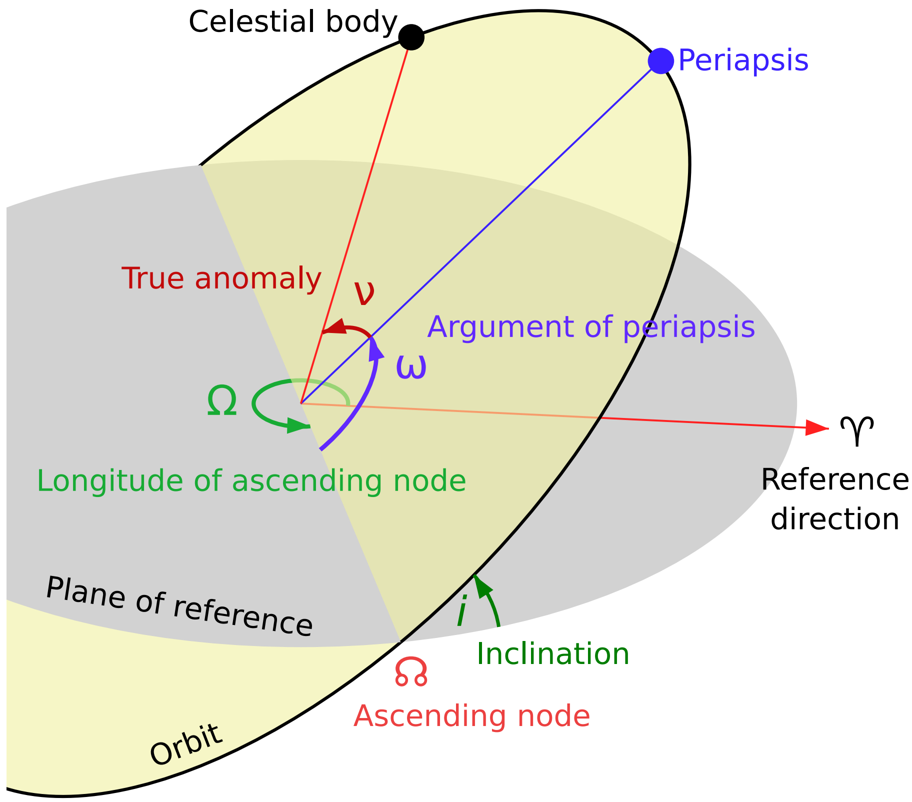


Figure 3.5: Three angles orient the periapsis in space, and a fourth gives the current position of the object along its orbit. [6]

$$E = \underbrace{\frac{1}{2}mv^2}_{\text{kinetic}} \underbrace{-\frac{\mathcal{G}Mm}{r}}_{\text{potential}} \text{ is constant} \tag{3.1}$$

Let us consider this at the two most notable points of a closed orbit: periapsis and apoapsis. At these points, the distance from center r is extremal (and so is the altitude, or distance from surface); thus, $\dot{r} = 0$ in both situations. It comes that:

$$v = \left|\dot{\vec{r}}\right| = \left|\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right| = \left|r\dot{\theta}\hat{\theta}\right| = r\dot{\theta}$$

Since we already know that $r^2\dot{\theta}$ is a constant value, it comes that:

$$r_a v_a = r_p v_p$$

Now, we apply the equation of energy (3.1) at both apses and get:

$$\begin{aligned}
\frac{1}{2}m{v_a}^2 - \frac{\mathcal{G}Mm}{r_a} &= \frac{1}{2}m{v_p}^2 - \frac{\mathcal{G}Mm}{r_p} \\
{v_a}^2 - {v_r}^2 &= 2\mathcal{G}M\left(\frac{1}{r_p} - \frac{1}{r_a}\right) \\
\left(1 - \frac{{r_a}^2}{{r_p}^2}\right){v_a}^2 &= 2\mathcal{G}M\left(\frac{1}{r_p} - \frac{1}{r_a}\right) \\
{v_a}^2 &= 2\mathcal{G}M\left(\frac{1}{r_p} - \frac{1}{r_a}\right)\frac{{r_p}^2}{{r_p}^2 - {r_a}^2} \\
&= 2\mathcal{G}M\frac{r_a - r_p}{r_p r_a}\frac{{r_p}^2}{{r_p}^2 - {r_a}^2} \\
&= 2\mathcal{G}M\frac{1}{r_a}\frac{r_p}{r_p + r_a} \\
&= \mathcal{G}M\frac{2a - r_a}{r_a a} \\
&= \mathcal{G}M\left(\frac{2}{r_a} - \frac{1}{a}\right)
\end{aligned}$$

We now have an expression of v_a that depends only on r_a and a. If we use the equation of energy (3.1) again on the with an arbitrary point with distance r and speed v:

$$\begin{aligned}
\frac{1}{2}m{v_a}^2 - \frac{\mathcal{G}Mm}{r_a} &= \frac{1}{2}mv^2 - \frac{\mathcal{G}Mm}{r} \\
v^2 &= \mathcal{G}M\left(\frac{2}{r} - \frac{2}{r_a}\right) + {v_a}^2 \\
v^2 &= \mathcal{G}M\left(\frac{2}{r} - \frac{1}{a}\right) \qquad (3.2)
\end{aligned}$$

From this, we can easily determine the speed of an object from its distance to the primary.

3.4 State prediction

Finding the position from the orbital parameters if relatively easy. First, we can find the true anomaly θ from the mean anomaly M. Then we can retrieve the distance from the trajectory equation:

$$r = \frac{l}{1 - e\cos(\theta)}$$

For this, we need l, that we can easily find back using the fact that $r_{\text{per}} = \frac{l}{1+e}$. With this, we have the polar coordinates (r, θ) in the oriented orbital plane.

Then, it is only a matter of rotating around the proper axes (ω around (z), i around (x) and Ω around (z)) to find the coordinates in the general frame of reference

Finding the velocity is very similar. We simply use the *vis viva* equation to determine the speed.

3.5 Inferring an orbit

We consider the inverse problem, where we want to determine the orbital parameters from our current position $\vec{r}$ and velocity $\vec{v}$.

First, let us define the **specific relative angular momentum** $\vec{h} = \vec{r} \times \vec{v}$. Since both $\vec{r}$ and $\vec{v}$ are contained in the orbital plane and $\vec{h}$ is orthogonal to both these vectors, it means that $\vec{h}$ is a **normal vector** of the orbital plane. The angle between $\vec{h}$ and (z) is the inclination i.

If we now compute $\vec{n} = \vec{u_z} \times \vec{h}$, we get a vector that is both in the (xOy) plane and in the orbital plane: this is the line of nodes. The angle between $\vec{n}$ and (x) is thus the longitude of ascending node, Ω.

The **eccentricity vector** is defined as:

$$\vec{e} = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{r}$$

It can be shown that its magnitude is the eccentricity and that it points towards the periapsis. We can thus compute $e = |\vec{e}|$, set ω as the angle between $\vec{n}$ and $\vec{e}$, and θ as the angle between $\vec{e}$ and $\vec{r}$.

It can be shown that $l = \frac{h^2}{\mu}$. Since we know the semi-latus rectum l and the eccentricity e, we know the periapsis $r_{\text{per}} = \frac{l}{1+e}$.

Chapter 4

Launch to orbit

> “Can it be that you have come from outer space?”
>
> – Yuri Gagarin, first human being in space, to people near his landing site

4.1 Vertical ascent

4.1.1 General expression

Assume that $\dot{m}$ is constant. Using (2.4), it comes that:

$$\Delta v = -v_e \ln\left(1 - \frac{\dot{m}}{m(0)} t\right) + I_F(t)$$

And integrating once more:

$$\begin{aligned} z &= -v_e \times \frac{1}{-\frac{\dot{m}}{m(0)}} \left(\left(1 - \frac{\dot{m}}{m(0)} t\right) \ln\left(1 - \frac{\dot{m}}{m(0)} t\right) + \frac{\dot{m}}{m(0)} t\right) + \underbrace{\int_0^t I_F(t) \mathrm{d}t}_{J_F(t)} \\ &= v_e \left(\frac{m(0)}{\dot{m}} - t\right) \ln\left(1 - \frac{\dot{m}}{m(0)} t\right) + v_e t + J_F(t) \end{aligned}$$

4.1.2 Gravity

If we assume $z \ll R$, we can estimate the effect of gravity $F = -\frac{\mathcal{G}Mm}{(R+z)^2} \simeq mg$ and $J_{mg} = \frac{1}{2} g t^2$.

The approximation is close enough for large bodies such as Earth or even Kerbin. On smaller bodies, there is no atmosphere and the ascent is not a problem. To verify this however, we will use (2.4) to get:

$$\ddot{z} = -v_e \frac{\dot{m}}{m(0) - \dot{m}t} - \frac{\mathcal{G}M}{(R+z)^2}$$

We will then proceed to a numerical approximation:

$$\begin{cases} z(0) = 0 \\ \dot{z}(0) = 0 \\ \ddot{z}(t) = -v_e \dfrac{\dot{m}}{m(0) - \dot{m}t} - \dfrac{\mathcal{G}M}{(R+z(t))^2} \\ \dot{z}(t+\mathrm{d}t) = \dot{z}(t) + \mathrm{d}t\ddot{z}(t) \\ z(t+\mathrm{d}t) = z(t) + \mathrm{d}t\dot{z}(t) \end{cases}$$

4.1.3 Graphs

We now compare the graphs for the different approaches (ignoring gravity, constant gravity, or numerical approximation). To get actual values, we simulate a basic rocket made of the following parts:

- Command Pod Mk1
- FL-T800 Fuel Tank
- LV-T30 Liquid Fuel Engine

?

We could see more clearly the contrast between the different situations by using a large pod (Mk1-2 Command Pod) instead. However, such a rocket will not go as high.

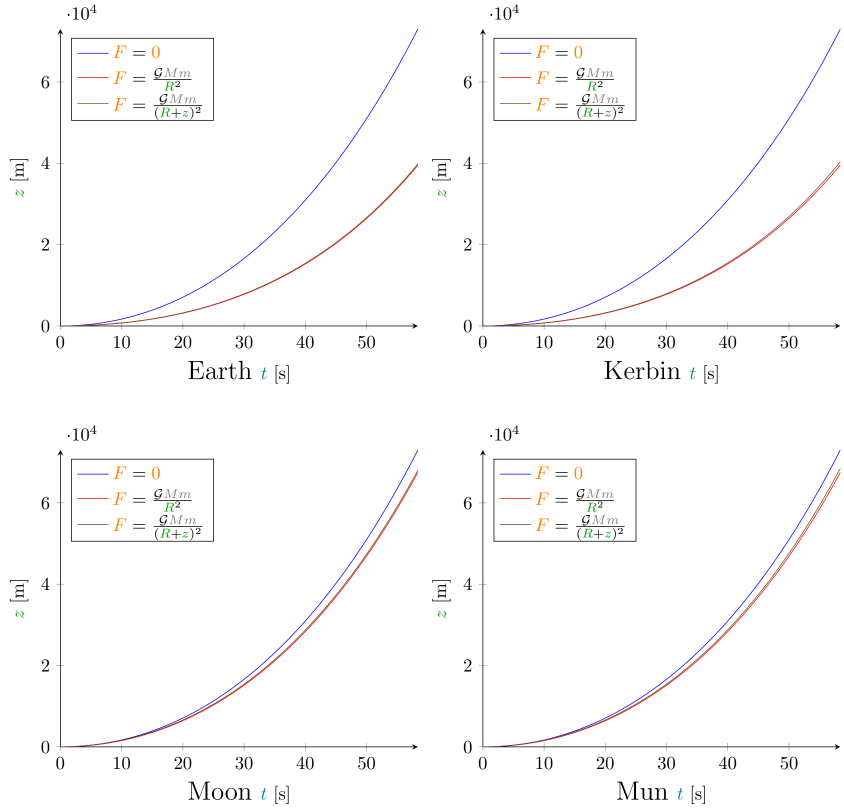


Figure 4.1: We can see that the assumption that $z \ll R$ is safe enough; moreover, the effect of gravity on small bodies (c) (d) is negligible on such short periods of time.

4.2 Atmospheric drag

4.2.1 Definition

Atmospheric drag exerts a force opposite to the velocity with intensity:

$$F_D = \frac{1}{2}\rho v^2 C_d A$$

where ρ is the density of air and C_d and A are aerodynamics factors.

4.2.2 Terminal velocity

When terminal velocity is reached, $\dot{z}$ is constant so $\ddot{z}$ is null and:

$$F_D = mg \Leftrightarrow v = \sqrt{2\frac{mg}{\rho C_d A}}$$

4.2.3 Optimal speed

Equation (2.4) shows us that if we want to optimize fuel consumption, we can simply reason on Δv.

$$\Delta v = (F_D + mg)\Delta t = \left(\frac{1}{2}\rho v^2 C_d A + mg\right)\frac{\Delta z}{v} = \left(\underbrace{\frac{1}{2}\rho C_d A}_{C_1} v + \underbrace{mg}_{C_2}\frac{1}{v}\right)\Delta z$$

Now we define $f(v) = C_1 v + \frac{C_2}{v}$ and find its minimum. Its derivative is $f'(v) = C_1 - \frac{C_2}{v^2}$. It comes that the minimum is $v = \sqrt{\frac{C_2}{C_1}}$. It matches the terminal velocity.

4.2.4 Derivation of air pressure

Assume the atmosphere is in a stable state and consider an infenitesimal volume $\mathrm{d}V$. The force exerted on it are the gravity and the pressure surrounding it. By symmetry, the effects of horizontal pressure cancels out; we note $\mathrm{d}P$ the difference in pressure below and above $\mathrm{d}V$.

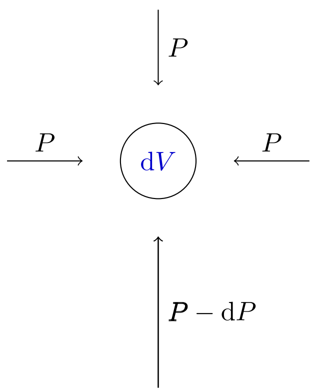


Figure 4.2: The pressure below $\mathrm{d}V$ must be stronger to fight gravity. $\mathrm{d}P$ will be a negative value, so that we can keep the vertical axis orientation up

The resultant force of pressure on $\mathrm{d}V$ is $\mathrm{d}A\mathrm{d}P$ where $\mathrm{d}A$ is the bottom area of $\mathrm{d}V$. We have $\mathrm{d}V = \mathrm{d}A\mathrm{d}z$. To fight gravity, we must have $\mathrm{d}A\mathrm{d}P = -g\mathrm{d}m = -g\rho\mathrm{d}V$. We can divide by $\mathrm{d}V$ on both hands of the equations to get:

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\rho g$$

The value g can assumed to be a constant but ρ depends on P. According to the ideal gas law:

$$PV = nRT \Rightarrow \rho = \frac{nM}{V} = \frac{PM}{RT}$$

where V is the considered volume, n the number of molecules of gas in it, R a convenient constant, T the temperature and M the mean mass of a molecule of the gas. From this, we can deduce the following differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}z} = -\frac{gM}{RT}P$$

To solve it, we need to find a primitive. We will assume that the gravitation is constant and that the temprature decrease linearly with altitude (rate L).

$$\int_0^z \frac{gM}{RT}\mathrm{d}z = \frac{gM}{R}\int_0^z \frac{1}{T - Lz}\mathrm{d}z = -\frac{gM}{LR}\left[\ln(T - Lz)\right]_0^z = -\frac{gM}{LR}\ln\left(1 - \frac{Lz}{T}\right)$$

Thus:

$$P(z) = P_0 e^{\frac{gM}{LR}\ln\left(1 - \frac{Lz}{T}\right)} = P_0\left(1 - \frac{Lz}{T}\right)^{-\frac{gM}{LR}}$$

4.2.5 Evolution of terminal velocity

Using the closed expression for the pressure, we can now derive the density and then the terminal velocity depending on the altitude:

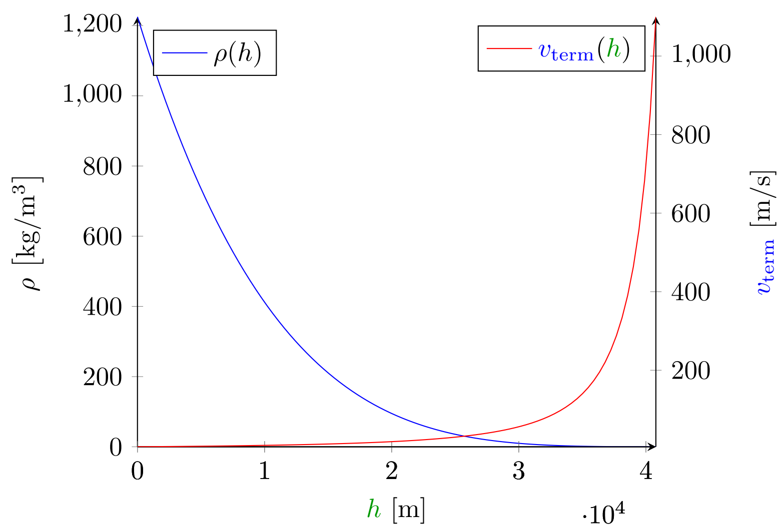


Figure 4.3: The terminal velocity quickly rises after 30 km

4.3 Gravity turn

4.3.1 Principle

Once the spacecraft has left the atmosphere, it is not subject to air drag anymore and can set its orbit. For this, it simply need to raise its orbit from the center of the body to above the atmosphere. It is more convenient for further maneuvers to have a circular orbit.

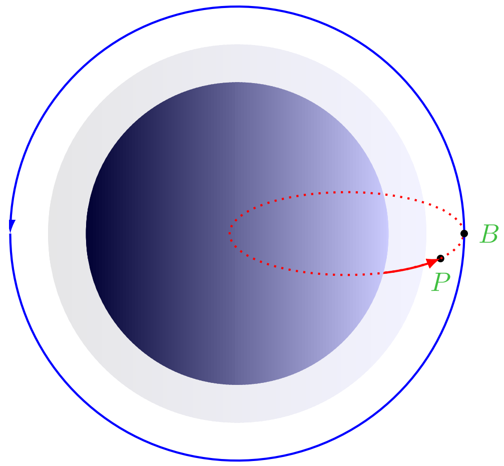


Figure 4.4: When P leaves the atmosphere and burns in B, it can change from the red "orbit" to the blue orbit.

4.3.2 Classical burn

Using the *vis-viva* equation (3.2), we can compute the requisite speed for orbiting:

$$v^2 = \mathcal{G}M\left(\frac{2}{r_a} - \frac{2}{r_a + r_a}\right) = \frac{\mathcal{G}M}{r_a}$$

For example, on Kerbin, you need to reach an horizontal velocity of:

$$v_{\text{orbit}} = \sqrt{\frac{6.67 \times 10^{11}\ m^3/kg/s^2 \times 5.29 \times 10^{22}\ kg}{600\ km + 80\ km}} = 2279\ m/s$$

To save Δv, launches are done close to the equator to go with the movement of the surface due to the body's rotation. On Kerbin, we save (T is the orbital period):

$$v_{\text{surf}} = \frac{2\pi R}{T} = 174.5\ m/s$$

Chapter 5

Maneuvers

> “The six words you *never* say at NASA: “And besides — it works in Kerbal Space Program.””
>
> – xkcd, cartoonist and NASA roboticist

In this chapter, we will see how to change from a circular orbit to another.

5.1 Pro-/retro- grade burn

The vis-viva equation (3.2) tells us:

$$v^2 = \mathcal{G}M \left(\frac{2}{r} - \frac{2}{r_a + r_p} \right)$$

For example, if we are at an apsis (apo- or peri-) and want to rise the opposite point, we need to speed up (burn prograde), and slow down to decrease it; the formula above tells us how much so.

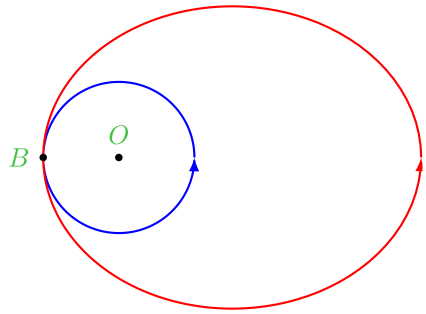


Figure 5.1: A satellite on the blue orbit can switch to the red one by burning prograde (speeding up) at B; conversely it can switch from the red orbit to the blue one by burning retro grade at this same point.

When searching for good trajectories, we are interested in saving propellant. According to (2.4), this is the same as saving for Δv (althgouh proportionally). If r is the apsis where the burn is performed, r_0 the opposite apsis before the burn and r_1 after:

$$
\begin{aligned}
\Delta v &= |v_1 - v_0| \\
\frac{1}{\sqrt{\mathcal{G}M}}\Delta v &= \left|\sqrt{\frac{2}{r} - \frac{2}{r + r_1}} - \sqrt{\frac{2}{r} - \frac{2}{r + r_0}}\right|
\end{aligned}
\tag{5.1}
$$

5.2 Hohmann transfer

Now, assume we are in a circular orbit of radius r_0 and want to do a simple transfer to a circular orbit of radius r_1. During a Hohmann transfer, we first raise our apoapsis to r_1 and then the periapsis (from the new apopsis).

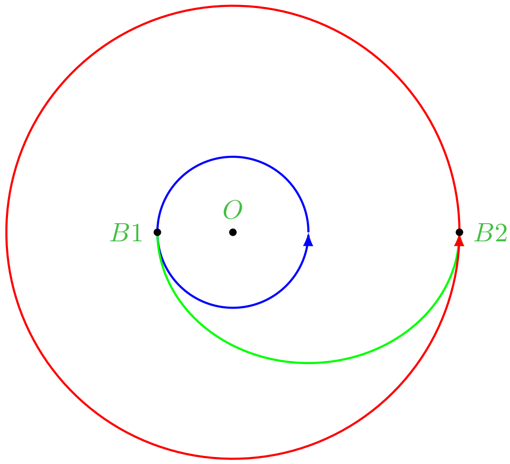


Figure 5.2: We first switch from the blue orbit to the green one by burning at $B1$ and then from the green one to the red one by burning at $B2$.

$$\frac{1}{\sqrt{\mathcal{G}M}}\Delta v = \left|\sqrt{\frac{2}{r_0} - \frac{2}{r_0 + r_1}} - \frac{1}{\sqrt{r_0}}\right| + \left|\frac{1}{\sqrt{r_1}} - \sqrt{\frac{2}{r_1} - \frac{2}{r_0 + r_1}}\right|$$

We can multiply both hands by $\sqrt{r_0}$ and set $x = \frac{r_1}{r_0}$ to get a simpler expression:

$$\underbrace{\sqrt{\frac{r_0}{\mathcal{G}M}}}_{\alpha}\Delta v = \left|\sqrt{2 - \frac{2}{1+x}} - 1\right| + \left|\frac{1}{\sqrt{x}} - \sqrt{\frac{2}{x} - \frac{2}{1+x}}\right|$$

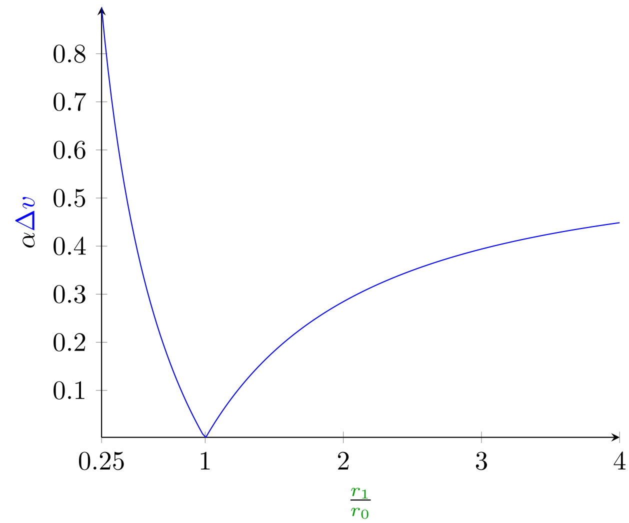


Figure 5.3: Note that decreasing an orbit by half costs about as much as the converse (doubling it), but dividing it by four costs twice as much as the converse.

5.3 Bi-elliptical transfer

The idea is to use three burns instead of two.

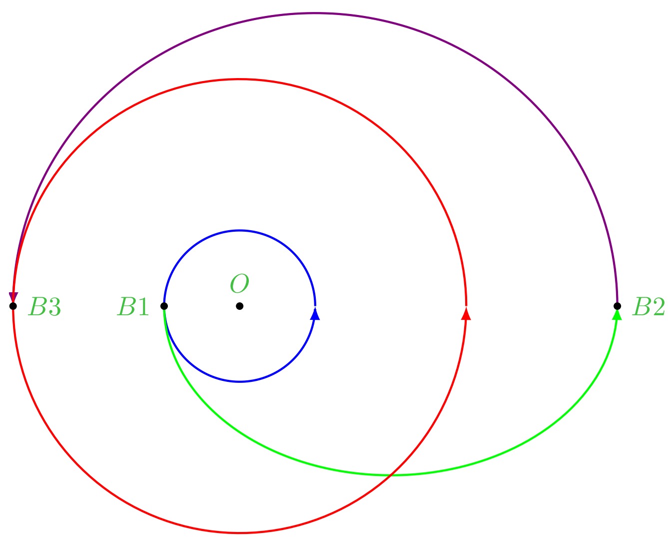
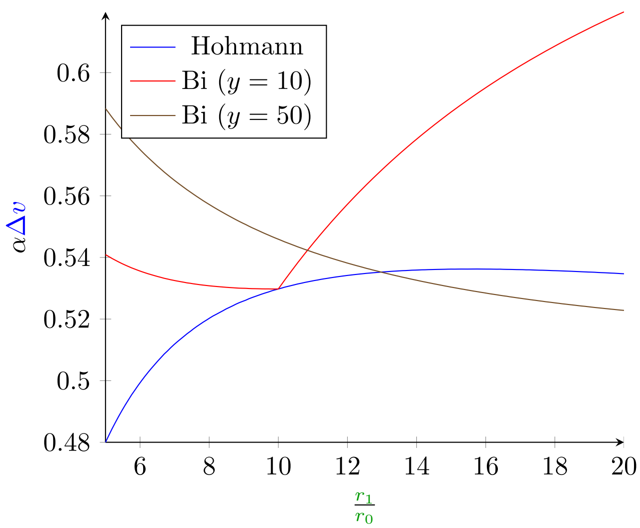


Figure 5.4: During a bi-elliptical transfer, we use two intermediate orbits (green, then violet); the idea is that it will be easier to raise the green periapsis from a higher apoapsis

$$
\begin{aligned}
\frac{1}{\sqrt{\mathcal{G}M}}\Delta v = &\left|\sqrt{\frac{2}{r_0} - \frac{2}{r_0 + r_1}} - \frac{1}{\sqrt{r_0}}\right| \\
&+ \left|\sqrt{\frac{2}{r_1} - \frac{2}{r_2 + r_1}} - \sqrt{\frac{2}{r_1} - \frac{2}{r_0 + r_1}}\right| \\
&+ \left|\frac{1}{\sqrt{r_2}} - \sqrt{\frac{2}{r_2} - \frac{2}{r_2 + r_1}}\right|
\end{aligned}
$$

Again, we set $x = \frac{r_2}{r_0}$ and $y = \frac{r_1}{r_0}$ and:

$$
\begin{aligned}
\underbrace{\sqrt{\frac{r_0}{\mathcal{G}M}}}_{\alpha}\Delta v = &\left|\sqrt{2 - \frac{2}{1 + y}} - 1\right| \\
&+ \left|\sqrt{\frac{2}{y} - \frac{2}{x + y}} - \sqrt{\frac{2}{y} - \frac{2}{1 + y}}\right| \\
&+ \left|\frac{1}{\sqrt{x}} - \sqrt{\frac{2}{x} - \frac{2}{x + y}}\right|
\end{aligned}
$$



5.4 Inclination change

! Remember, we are only considering **circular** orbits. The formulas and derivations below only make sense for circular orbits. We advise you to set

your inclination in a circular orbit before any subsequent maneuver.

5.4.1 (Anti-)normal burn

Consider the orbital plane in which a satellite is moving. We are interested in the effect of an acceleration orthogonal to the plane (normal or antinormal). For this, we study the evolution of the velocity.

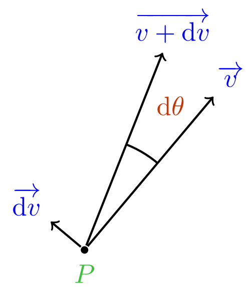


Figure 5.5: The satellite is heading towards $\vec{v}$ and an acceleration is applied to it so that during a time $\mathrm{d}t$, its velocity is changed by $\vec{\mathrm{d}v}$.

As seen in figure (5.5), we can easily find the change in inclination:

$$\begin{aligned}
\mathrm{d}\theta \simeq \tan \mathrm{d}\theta &= \frac{\mathrm{d}v}{v} \\
\int \frac{\mathrm{d}\theta}{\mathrm{d}t}\mathrm{d}t &= \int \frac{a\mathrm{d}t}{v} \\
\theta &= \frac{at}{v} = \frac{\Delta v}{v}
\end{aligned}$$

5.4.2 Straight burn

Doing a 180° inclination chage using a constant radial burn like exposed above yields a Δv proportional to the current orbital velocity v: $\Delta v = \pi v \simeq 3v$. However, simply going retrograde until the speed is reverse only yields $\Delta v = 2v$ for the same result.

You can find another derivation of the cost in [9] [10].

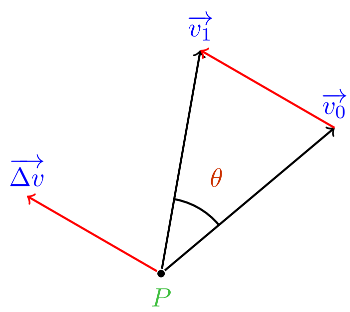


Figure 5.6: A rotation of angle θ from velocity vector $\vec{v_0}$ to $\vec{v_1}$ is done in a straight change in speed Δv.

We need to compute Δv for given $v = |\vec{v_0}| = |\vec{v_1}|$ and θ. Because the triangle is isosceles, the altitude and the median from P are one so:

$$\Delta v = \left| 2v \sin \frac{\theta}{2} \right|$$

For $\theta = \pi$, we get $\Delta v = 2v$ which is the expected result.

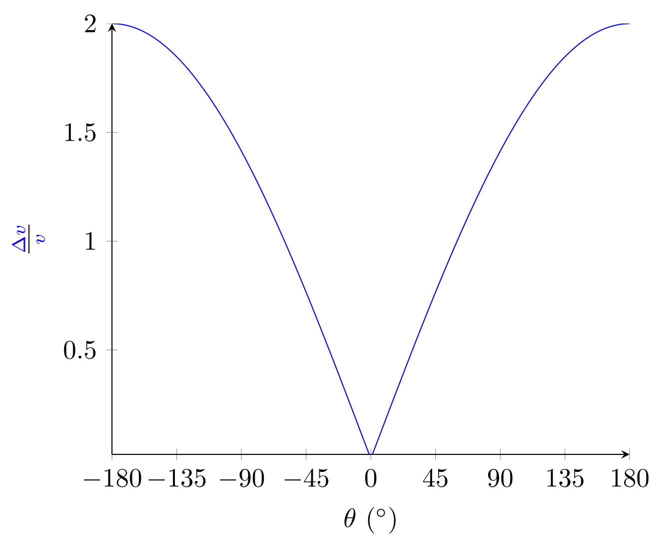


Figure 5.7: An inclination change of about $30°$ already costs half the orbital speed; $60°$ costs as much as the orbital speed.

5.4.3 Bi-elliptical inclination change

Whatever the method used for the inclination change, the cost is proportional to the current orbital speed. Thus, it is more efficient to do such a maneuver at low speed (e.g. at apoapsis). /u/ObsessedWithKSP demonstrated a maneuver similar to the bi-elliptical transfer for a more efficient plane change [7] [8].

A formal derivation of the optimal inclination change has been published by /u/listens_to_galaxies [9] [10].

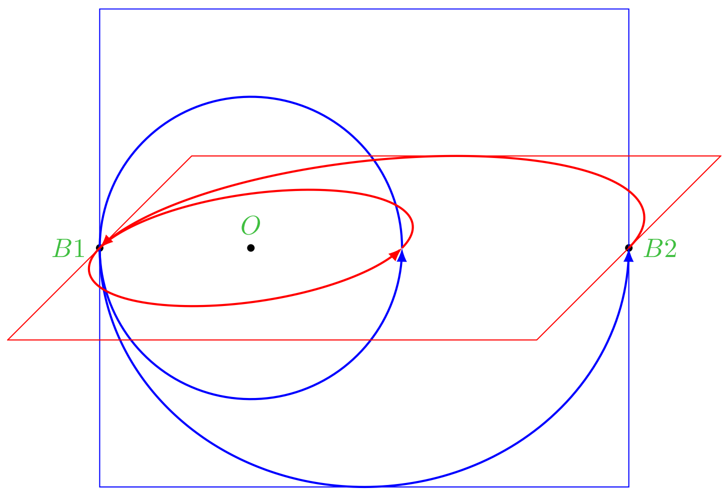


Figure 5.8: Starting in the blue plane on the circular orbit, the spacecraft first burns prograde in $B1$ to raise its apoapsis to $B2$; once there, its speed is lower and it can proceed to the inclination change to the red plane effectively; finaly, it burns retrograde back in $B1$ to return to a circular orbit.

5.5 Radial in/out burn

TODO

5.6 Arbitrary burn

TODO

Chapter 6

Operating

"Let's face it, space is a risky business. I always considered every launch a barely controlled explosion."

– Aaron Cohen, Deputy Administrator of NASA

6.1 Gravity assist

Consider a spacecraft P and a celestial body O. When getting close to O, the gravity will get important enough to significantly change the trajectory of the spacecraft. The goal is to get it to turn without using any fuel and while keeping the same velocity (in a different direction).

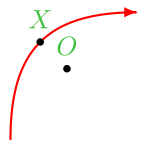


Figure 6.1: The speed of the mobile will increase as it heads towards the periapsis X, trading its gravitational potential for kinetic energy; it will then decrease when moving away from X. Overall, the total energy will be conserved and it will move as fast when leaving O as when getting there.

Now, this velocity change is done relatively to O.

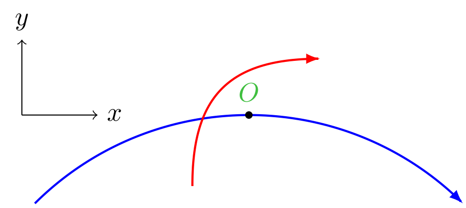


Figure 6.2: If we note v_P the velocity of P relatively to O and v_O the velocity of O around its primary, the velocity of P around O's primary is v_P before encountering O and $v_P + v_O$ after (note that the red trajectory moves with O along the blue trajectory).

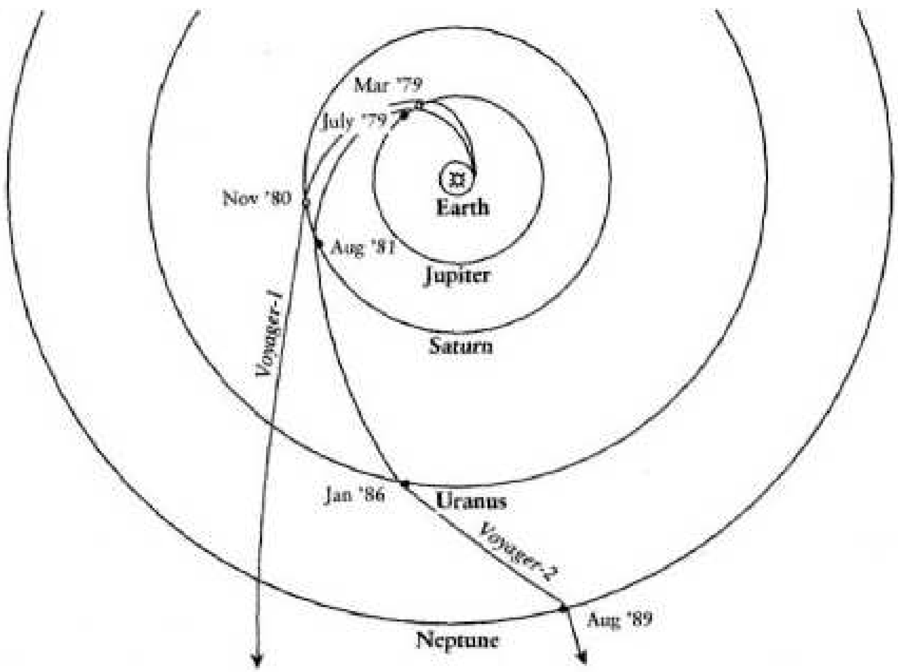


Figure 6.3: Voyager 2 flew by Jupiter, Saturn, Uranus and Neptune, but the swingby of Neptune actually slowed down the vessel as it turned slightly out of the trajectory (out of the plane in the image). [17]

6.2 Rendez-vous

The goal of rendez-vous is to make the position **and velocity** of a spacecraft $P1$ roughly matching those of $P2$. Once the spacecrafts have performed a rendez-vous, they can do useful action such as docking or crew transfer.

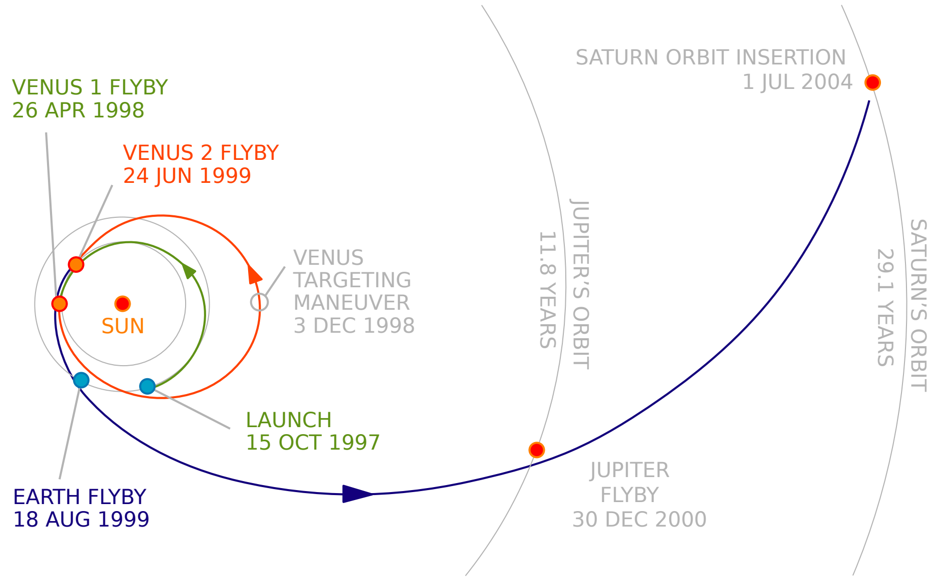


Figure 6.4: The Cassini-Huygens spacecraft left the Earth for the green solar orbit (1997); it then flew twice by Venus to increase its orbit to the red one (1998) and then to the blue one (1999). The blue trajectory was further improved by gravity assists by the Earth (1999) and by Jupiter (2000). [18]

? After rendez-vous, two vessels would be moving along a very similar orbit. On short durations (compared to the orbital periods), the pull of gravity can simply be ignored like if the vessels were subject to no force at all (making docking simple).

Same position and same velocity means a similar orbit. Once both satellites are revolving around the same body, the first correction is that of inclination: one will perform an inclination change (either (anti-)normal burn, straight burn, or bi-elliptical). When they are orbiting in the same plane, they will have to consort their orbits, and synchronize on the final orbit.

A simple way to do this is to have the orbits joining in one point (say, the periapsis) while their period is significantly different. After a few revolutions, the two spacecraft will get in the common point at the same time and will be able to join their orbit. It is handy to correct the orbits before the last revolution to get a more accurate rendez-vous.

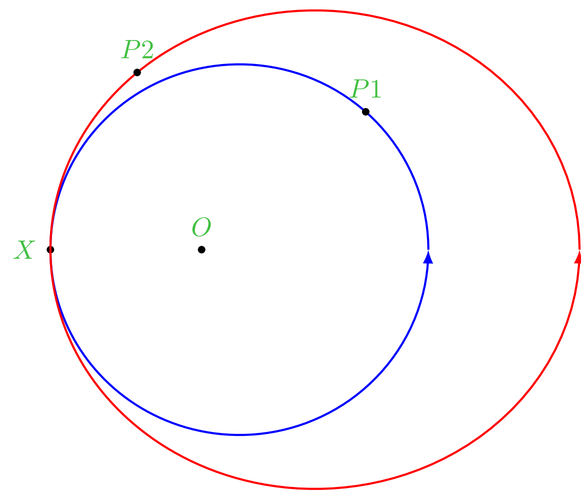


Figure 6.5: $P2$ will reach X ahead of $P1$; however, the red orbit has a higher apoapsis and thus a longer period: after some time, $P1$ will catch up with $P2$

6.3 Satellite coverage

Assume we want to set up a satellite network around a body O. The coverage is done with satellites in circular orbits. It is handy to have several satellites around the same orbit; to cover a band on the surface at all time. The fist thing to ensure is that the satellites stays connected.

6.3.1 Connectivity

We are interested in two values: the closest approach of the lines-of-sight to O, notated h, that must be greater than the radius of the body R and the distance between the satellites, notated s that must be lower than the range of the satellites antennas r_a.

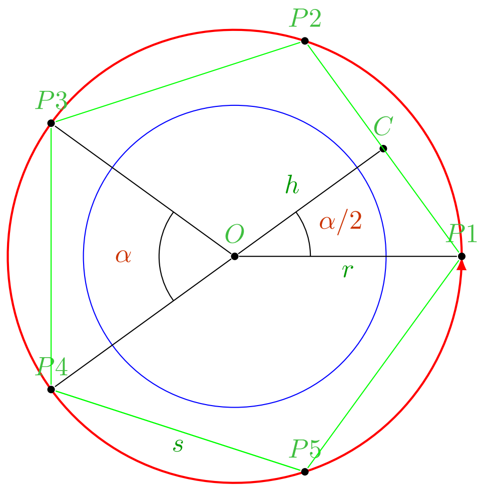


Figure 6.6: There are 5 satellites on the red orbit around O. The surface is shown in blue and the lines-of-sight of the satellites in green.

What we have to choose is the number of satellites n and their distance from O, r. With, $\alpha = \frac{2\pi}{n}$ and simple trigonometry, we get that:

$$h = r\cos\frac{\alpha}{2} = r\cos\frac{\pi}{n}$$

and

$$s = 2r\sin\frac{\alpha}{2} = 2r\sin\frac{\pi}{n}$$

Thus, with the requirement that $h > R$ and $s > r_a$, we need:

$$\frac{R}{\cos\frac{\pi}{n}} < r < \frac{r_a}{2\sin\frac{\pi}{n}}$$

Conversely, if we want to know the the necessary value of n for a given altitude r, we have to ensure that:

$$n \geq \frac{\pi}{\arccos\frac{R}{r}} \text{ and } n \geq \frac{\pi}{\arcsin\frac{r_a}{2r}}$$

? Functions arccos and arcsin are only defined from -1 to 1 meaning that we need that $r > R$; this makes sense. Moreover, when $2r < r_a$, the second conditions should be ignored, since it means the whole orbit stays in range.

Similar information is available on the KSP wiki [12].

6.3.2 Coverage

The curvature of Kerbin makes that a satellite sees less than half the surface; the closer it is the surface, the less it can see.

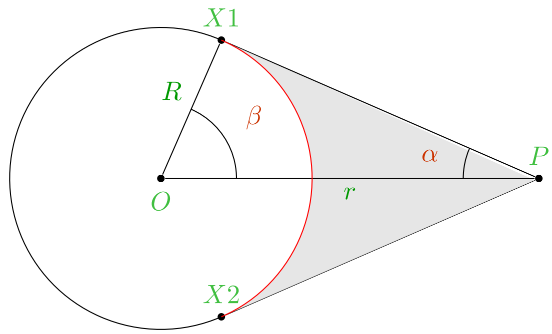


Figure 6.7: $X1$ and $X2$ are the farthest point that P can see from this distance; the line $(PX1)$ is said to be tangent to the circle, and there is a right angle in $X1$

This property is used on the surface: the crow's nest is an observation spot located high in the masts of a ship to see significantly farther away. The distance you can see from altitude a is $R \arccos \frac{R}{R+a}$.

We quickly see that $\sin \alpha = \cos \beta = \frac{R}{r}$.

6.4 Light exposition

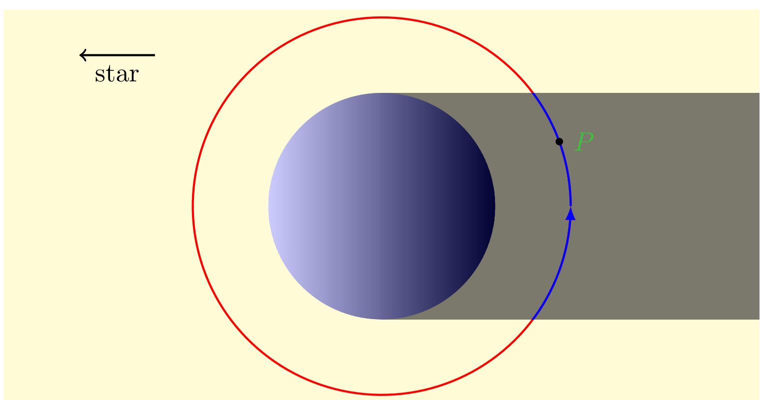


Figure 6.8: The light rays are shown in orange. When P is in the blue part of the orbit, it does not receive light from the primary star.

> ! If O is a moon orbiting around a planet, both the night time around the moon and around the planet must be computed. In the worst case (when the night of one ends, the night of the other starts), they can add up and make a longer night than expected.

6.5 Flight duration

TODO

Bibliography

[1] "Kerbal Space Program",
2011–2014, early access game, Squad,
https://www.kerbalspaceprogram.com/

[2] "Three Kerbal Mun"
2011, deviantART publication, MK01,
https://mk01.deviantart.com/art/Three-Kerbal-Mun-272960572

[3] "Renderings of my favourite planets/muns in banner form",
2013, KSP forum post, Lunniy Korabl,
http://forum.kerbalspaceprogram.com/threads/34329

[4] "The Six Words You Never Say in KSP",
2013, KSP forum post, Hayoo,
http://forum.kerbalspaceprogram.com/threads/50660?p=764793#post764793

[5] "Rareden's Projects",
2013, KSP forum post, Rareden,
http://forum.kerbalspaceprogram.com/threads/25231

[6] Orbit angles,
2010, SVG diagram, User:Theoprakt,
https://en.wikipedia.org/wiki/File:Orbit1.svg

[7] "How to do a bi-elliptic inclination change transfer orbit in one picture.",
2014, Reddit submission, /u/ObsessedWithKSP,
https://pay.reddit.com/r/KerbalSpaceProgram/comments/23ri7a

[8] "How to do a bi-elliptic inclination change transfer orbit in one picture.",
2014, Reddit submission, /u/ObsessedWithKSP,
https://pay.reddit.com/r/KerbalSpaceProgram/comments/23ri7a

[9] "[D]erivation of the [optimal] inclination change transfer orbit",
2014, Reddit submission, /u/listens_to_galaxies,
https://pay.reddit.com/r/KerbalAcademy/comments/23v5wo

[10] "Bi-elliptic inclination change transfer orbit",
2014, imgur album, /u/listens_to_galaxies,
https://imgur.com/a/uM7PK

[11] "Tutorial: Complete Novice's Guide to RemoteTech 2",
2013, Reddit submission, /u/Grays42,
https://pay.reddit.com/r/KerbalSpaceProgram/comments/1rml0r

[12] "Tutorial: Satellite Coverage",
2013–2014, KSP wiki article, XZise *et alii*,
http://wiki.kerbalspaceprogram.com/wiki/Tutorial:Satellite_Coverage

[13] "Kerbal Gravity Wells",
2013, Reddit submission, /u/jofwu,
https://pay.reddit.com/r/KerbalSpaceProgram/comments/1o3epz

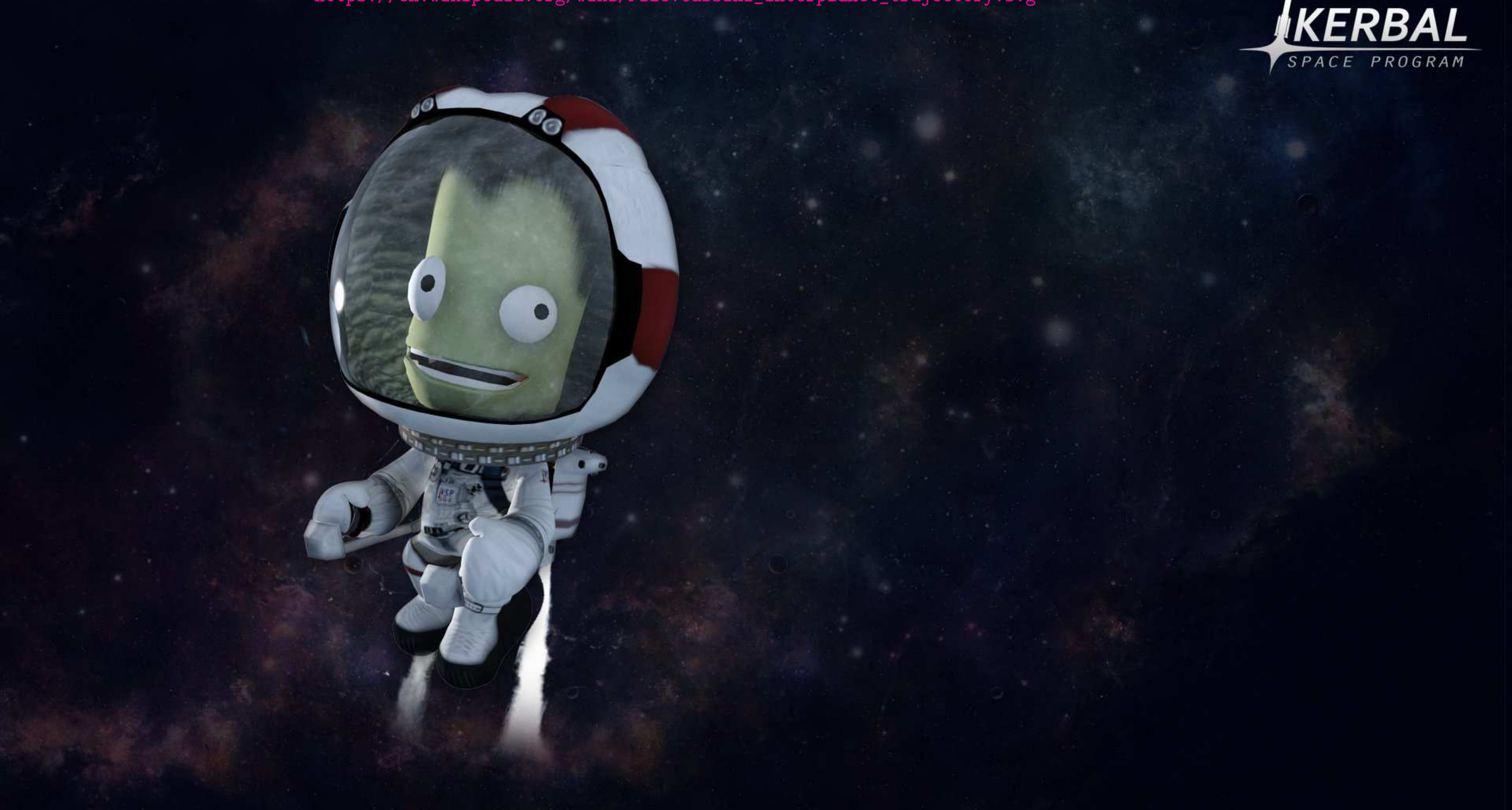
[14] "Six Words",
2013, webcomic, xkcd,
https://xkcd.com/1244/

[15] "Orbital Mechanics",
2014, webcomic, xkcd,
https://xkcd.com/1356/

[16] "KSP – Flags",
2013, Imgur album, /u/shard013,
https://imgur.com/a/rRUZp

[17] "Ken Lunn's Voyager Spacecraft Page",
2012, web page, Ken Lunn,
http://usuaris.tinet.org/klunn/voyager.html

[18] "Cassini interplanetary trajectory",
2006, SVG diagram, User:Pline,
https://en.wikipedia.org/wiki/File:Cassini_interplanet_trajectory.svg



“My goal is simple. It is a complete understanding of the universe, why it is as it is and why it exists at all.”

– Stephen Hawking